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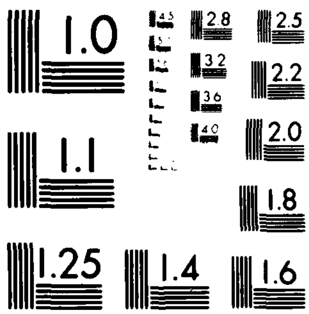
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FURTHER RESULTS ON THE STABILITY OF A THICK ELASTIC PLATE UNDER THRUST

by

K.N. Sawyers and R.S. Rivlin

TECHNICAL REPORT NO. CAM-100-38 April, 1982

OFFICE OF NAVAL RESEARCH CONTRACT NO. N00014-76-C-0235

Center for the Application of Mathematics (CAM)

Further Results on the Stability of a Thick Elastic Plate Under Thrust

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ABSTRACT

The stability of the critical state for flexural buckling of a plate of incompressible isotropic elastic material with arbitrary strain-energy function is studied. Most of the analysis is carried out without restriction on the magnitude of the aspect ratio of the plate. However, the final result is limited to the case when the aspect ratio is such that terms of fourth degree in it may be neglected in comparison with terms of zero degree.

1. Introduction

In the present paper we consider a plate of incompressible isotropic elastic material to be situated with its edges parallel to the axes of a rectangular cartesian coordinate system x . The plate is acted on by dead-loads applied normally to the faces of the plate which are perpendicular to the 1 and 3-axes of the system x . The faces perpendicular to the 2-axis are force-free. The constraints on the 1 and 3-faces are such as to permit the plate to undergo pure homogeneous deformations. It is supposed that the load in the 1-direction is a thrust. As this thrust is increased static bifurcation solutions in the 12-plane, superposed on a uniform extension in the 3-direction, become possible at certain critical values. These solutions may correspond to buckling of the plate of the flexural or barreling type. In [1,2] the compression ratios in the 1-direction at which these critical values of the thrust are reached was calculated for an arbitrary strain-energy function.

In a previous paper [3] we discussed the stability of the states of pure homogeneous deformation at which these bifurcations occur, with the assumption that the strain-energy function is neo-Hookean. The stability criterion employed and the procedure adopted was essentially that due to Koiter . An equilibrium state is regarded as stable or unstable accordingly as the potential energy of the system, consisting of the body and loads, has a proper minimum at this state with respect to all infinitesimal deformations satisfying the kinematic constraints. It was seen in [3] that a state of pure homogeneous strain for which a bifurcation solution exists is one of neutral equilibrium. The state is stable if the potential energy of the system is smaller for this state than it is for every state in its neighborhood which satisfies the kinematic constraints; otherwise it is unstable.

With this criterion it was shown in [3] that at critical compression ratios for flexural buckling the homogeneous state is stable provided that the aspect ratio (2-dimension/1-dimension) is less than about 0.2, and unstable otherwise. This implies that the immediate post-buckling behavior will be stable for the lower aspect ratios and of the snap-through type at the higher aspect ratios. At critical compression ratios for buckling of the barreling type, the homogeneous state is stable for all aspect ratios. These calculations were carried out in [3] for aspect ratios ranging from zero to infinity. The formula on which these calculations were based is extremely complicated and acquires meaning only as the result of the numerical computations. Accordingly, for the case of flexural buckling an asymptotic calculation valid for small values of the aspect ratio was also carried out. As the aspect ratio tends to zero this result agrees with the classical result of Euler based on the theory of the elastica.

In the present paper analogous calculations are carried out for an incompressible isotropic elastic material with arbitrary, rather than neo-Hookean, strain-energy function. While much of the analysis (up to §7) is carried out with no restrictions on the magnitude of the aspect ratio, the final result, which is given in equation (11.25), is obtained only for the case of small aspect ratio.

2. Statement of the problem

We consider a rectangular plate of incompressible isotropic elastic material, which has its edges parallel to the axes of a rectangular cartesian coordinate system x. Let $\underset{\sim}{\xi}$ be the vector position, relative to the origin of the system x, of a generic particle of the plate in its undeformed state (state 0) and let its bounding surfaces in this state be the planes

$$\xi_A = \pm \ell_A \quad (A=1,2,3) . \tag{2.1}$$

We suppose that the plate is maintained in an equilibrium state of pure homogeneous deformation (state I), with extension ratios $\lambda_1,\lambda_2,\lambda_3$ and principal directions parallel to the coordinate axes, by *uniformly distributed normal tractions* applied to the surfaces $\xi_1 = \pm \ell_1$ and $\xi_3 = \pm \ell_3$, the surfaces $\xi_2 = \pm \ell_2$ being force-free. Let Π_{11} and Π_{33} be the tractions, measured per unit undeformed area, applied to the faces $\xi_1 = \ell_1$ and $\xi_3 = \ell_3$ respectively.

We assume that the surfaces initially at $\xi_1 = \pm \ell_1$ and $\xi_3 = \pm \ell_3$ are constrained so that they move parallel to the 1 and 3-axes respectively, but points on them are free to move in the planes normal to these directions (i.e. the tangential tractions on these surfaces are assumed to be zero).

Let $\underset{\sim}{X}$ be the vector position in state I of the particle which has vector position $\underset{\sim}{\xi}$ in state 0.

Then,*

$$X_A = \lambda_A \xi_A \quad (A=1,2,3), \quad \lambda_1\lambda_2\lambda_3 = 1 . \tag{2.2}$$

Now suppose that the plate undergoes a further deformation which consists of a uniform stretch in the 3-direction and a plane deformation in the 12-plane. We call the resulting state of deformation of the plate state II, and we write

$$\underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{u} , \tag{2.3}$$

where

$$u_1 = u_1(\xi_1,\xi_2) , \quad u_2 = u_2(\xi_1,\xi_2) , \quad u_3 = \lambda_3 E \xi_3 , \tag{2.4}$$

and E is a constant.

Since the material is incompressible, $\det||x_{i,\alpha}|| = 1$. It follows with (2.2)-(2.4) that

$$(1+E)(\lambda_1 u_{2,2}+\lambda_2 u_{1,1}+u_{1,1}u_{2,2}-u_{1,2}u_{2,1}) + \lambda_1\lambda_2 E = 0 . \tag{2.5}$$

Let $\underset{\sim}{C} = ||C_{ij}||$, $\underset{\sim}{\gamma} = ||\gamma_{ij}||$ be the Finger strain matrices in states I and II respectively. Then, with (2.2),

$$C_{AB} = \lambda_A^2 \delta_{AB} , \quad \gamma_{ij} = x_{i,m}x_{j,m} . \tag{2.6}$$

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* Throughout this paper, latin subscripts take the values 1,2,3 and greek subscripts take the values 1,2. The Einstein summation convention applies to lower case latin and greek subscripts but not to upper case latin subscripts. Also, the operators $\partial/\partial\xi_\alpha$, $\partial/\partial\xi_m$ are denoted $,\alpha$ and $,m$ respectively.

We introduce the notation

$$\underset{\sim}{\gamma} = \underset{\sim}{C} + \underset{\sim}{c} , \quad \underset{\sim}{c} = ||c_{ij}|| . \tag{2.7}$$

From (2.4), (2.6) and (2.7), we obtain

$$\begin{aligned} &c_{11} = (2\lambda_1+u_{1,1})u_{1,1}+u_{1,2}^2 , \quad c_{22} = (2\lambda_2+u_{2,2})u_{2,2}+u_{2,1}^2 , \\ &c_{33} = \lambda_3^2 E(2+E) , \; c_{12} = c_{21} = (\lambda_1+u_{1,1})u_{2,1}+(\lambda_2+u_{2,2})u_{1,2}, \\ &c_{ij} = 0 \quad (ij=23,32,13,31) . \end{aligned} \tag{2.8}$$

Let ι_1, ι_2 be the invariants of $\underset{\sim}{\gamma}$ and I_1, I_2 those of $\underset{\sim}{C}$ defined by

$$\begin{aligned} &I_1 = \lambda_i\lambda_i , \quad I_2 = \lambda_i^{-1}\lambda_i^{-1}, \\ &\iota_1 = \mathrm{tr}\,\underset{\sim}{\gamma} = I_1+i_1 , \; \iota_2 = \tfrac{1}{2}\{(\mathrm{tr}\,\underset{\sim}{\gamma})^2-\mathrm{tr}\,\underset{\sim}{\gamma}^2\} = I_2+ i_2 . \end{aligned} \tag{2.9}$$

With the notation

$$\begin{aligned} &\lambda = \lambda_2/\lambda_1 , \\ &\kappa[\underset{\sim}{u}] = u_{\alpha,\beta}u_{\alpha,\beta} + 2\lambda(u_{1,2}u_{2,1}-u_{1,1}u_{2,2}) , \end{aligned} \tag{2.10}$$

it follows from (2.5)-(2.9) that

$$i_1 = i, \quad i_2 = \lambda_3^2(i+j) , \tag{2.11}$$

where

$$i = 2\{\lambda_1(1-\lambda^2)u_{1,1} + (\lambda_3^2-\lambda_2^2)E\} + \kappa[\underset{\sim}{u}]$$
$$+ (\lambda_3^2+2\lambda_2^2)E^2 - 2\lambda_2^2E^3(1+E)^{-1} ,$$
$$j = E\{2(\lambda_1^2+\lambda_2^2-\lambda_3^2-\lambda_1^4\lambda_2^4) + (2+E)i \tag{2.12}$$
$$+ E(\lambda_1^2+\lambda_2^2-5\lambda_3^2+3\lambda_1^4\lambda_2^4)$$
$$- E^2[\lambda_3^2(4+E) + \lambda_1^4\lambda_2^4(4+3E)(1+E)^{-2}]\} .$$

Let W and w denote the strain energies per unit volume in states I and II respectively. Then,

$$w = w(\iota_1,\iota_2) , \quad W = w(I_1,I_2) . \tag{2.13}$$

We introduce the notation

$$w_\alpha = \frac{\partial w}{\partial \iota_\alpha} , \quad w_{\alpha\beta} = \frac{\partial^2 w}{\partial \iota_\alpha \partial \iota_\beta} ,$$
$$W_\alpha = w_\alpha(I_1,I_2) , \quad W_{\alpha\beta} = w_{\alpha\beta}(I_1,I_2) . \tag{2.14}$$

The increase in the strain-energy of the plate in passing from state I to state II is given by*

$$2\ell_3 \int\int (w-W)d\xi_1 d\xi_2 = 2\ell_3 \int\int (W_\alpha i_\alpha + \tfrac{1}{2} W_{\alpha\beta} i_\alpha i_\beta$$
$$+ W^{(3)} + W^{(4)} + \bar{W}^{(5)})d\xi_1 d\xi_2 , \tag{2.15}$$

* The domain of integration for the double integrals is the rectangle $(-\ell_1,\ell_1)\times(-\ell_2,\ell_2)$.

where

$$W^{(3)} = \frac{1}{6} W_{\alpha\beta\gamma} i_\alpha i_\beta i_\gamma \ , \quad W^{(4)} = \frac{1}{24} W_{\alpha\beta\gamma\delta} i_\alpha i_\beta i_\gamma i_\delta \ , \tag{2.16}$$

and $\overline{W}^{(5)}$ denotes the usual Taylor series remainder.

Let $\Pi_{\alpha i}$ denote the Piola-Kirchhoff stress in state I . Then (cf.[1], equation (3.6))

$$\begin{aligned}
\Pi_{11} &= 2\lambda_1(1-\lambda^2)(W_1+\lambda_3^2 W_2) \ , \\
\Pi_{33} &= 2(\lambda_3-\lambda_2^2\lambda_3^{-1})(W_1+\lambda_1^2 W_2) \ , \\
\Pi_{\alpha i} &= 0 \quad (\alpha i \neq 11,33) \ .
\end{aligned} \tag{2.17}$$

It follows from the constraint conditions on the surfaces $\xi_1 = \pm\ell_1$ that the displacement field $\underset{\sim}{u}$ must satisfy the conditions

$$u_{1,2}(\pm\ell_1,\xi_2) = 0 \ . \tag{2.18}$$

With the further assumption that the displacements of these two surfaces are equal and opposite, we obtain

$$u_1(\ell_1,\xi_2) = -u_1(-\ell_1,\xi_2) = \lambda_1 e \ell_1 \ , \tag{2.19}$$

where e is a constant.

The resultant forces acting on the faces $\xi_1 = \pm\ell_1$ and $\xi_3 = \pm\ell_3$ in state I are obtained from (2.17) as $\pm 4\ell_2\ell_3\Pi_{11}$ and $\pm 4\ell_1\ell_2\Pi_{33}$ respectively. We suppose that state II is reached from state I under dead-loading conditions. Then these are also the resultant forces acting in state II. The increase in the potential energy of these forces in the passage from state I to state II is

$$-8\ell_1\ell_2\ell_3(\lambda_1 e\Pi_{11} + \lambda_3 E\Pi_{33}) \ . \tag{2.20}$$

With (2.17) and (2.19) this may be written in the form

$$-4\ell_3 \int\int\{\lambda_1(1-\lambda^2)(W_1+\lambda_3^2W_2)u_{1,1}+(\lambda_3^2-\lambda_2^2)(W_1+\lambda_1^2W_2)E\}d\xi_1 d\xi_2 . \tag{2.21}$$

The increase $G[\underset{\sim}{u}]$ in the total potential energy of the system, consisting of the body and loads, resulting from the deformation from state I to state II is given by the sum of the quantities in (2.15) and (2.21). With (2.11) and (2.12), $G[\underset{\sim}{u}]$ can be expressed by

$$\begin{aligned} G[\underset{\sim}{u}] = 2\ell_3 \int\int & [(W_1+\lambda_3^2W_2)\{\kappa[\underset{\sim}{u}] + (\lambda_3^2+2\lambda_2^2)E^2\} \\ & + \lambda_3^2W_2\{4\lambda_1(1-\lambda^2)Eu_{1,1} + (\lambda_1^2-\lambda_3^2+3\lambda_1^4\lambda_2^4-3\lambda_2^2)E^2 \\ & \qquad + 2E\kappa[u]\} \\ & + \tfrac{1}{2}\{W_{11}i^2+2\lambda_3^2W_{12}i(i+j) + \lambda_3^4W_{22}(i+j)^2\} \\ & + W^{(3)} + W^{(4)} - 2\lambda_2^2W_1E^3(1+E)^{-1} \\ & + \lambda_3^2W_2E^2\{i-E[\lambda_3^2(2+e) - 2\lambda_2^2(1+E)^{-1} \\ & + \lambda_1^4\lambda_2^4(4+3E)(1+E)^{-2}]\} + \overline{W}^{(5)}]d\xi_1 d\xi_2 \ . \end{aligned} \tag{2.22}$$

We shall say that the state I is stable if $G[\underset{\sim}{u}]$ is positive definite for all $\underset{\sim}{u}$ lying in a neighborhood of $\underset{\sim}{u} = \underset{\sim}{0}$ and satisfying the kinematic constraints (2.5) and (2.19).

3. Critical equilibrium states

A necessary condition for stability of state I is that the second variation $G_2[\underset{\sim}{u}]$ of $G[\underset{\sim}{u}]$ be non-negative for all sufficiently small, kinematically admissible values of $\underset{\sim}{u}$. With (2.10) and (2.12) it follows from (2.22) that

$$\begin{aligned} G_2[\underset{\sim}{u}] = 2\ell_3 \int\int [&k_1\{\kappa[\underset{\sim}{u}] + (\lambda_3^2+2\lambda_2^2)E^2\} \\ &+ \lambda_3^2 W_2\{4\lambda_1(1-\lambda^2)Eu_{1,1}+\Lambda_1 E^2\} \\ &+ k_2 k^2 + 2k_{21}\Lambda_2 Ek + 2\lambda_3^4 W_{22}\Lambda_2^2 E^2] d\xi_1 d\xi_2 \ , \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &k_1 = W_1+\lambda_3^2 W_2 \ , \quad k_2 = 2(W_{11}+2\lambda_3^2 W_{12}+\lambda_3^4 W_{22}) \ , \\ &k_{21} = 2\lambda_3^2(W_{12}+\lambda_3^2 W_{22}) \ , \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} k &= \lambda_1(1-\lambda^2)u_{1,1} + (\lambda_3^2-\lambda_2^2)E \ , \\ \Lambda_1 &= (\lambda_1^2-\lambda_3^2)(3\lambda_2^2+\lambda_3^2)\lambda_3^{-2} \ , \\ \Lambda_2 &= -(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\lambda_3^{-2} \ . \end{aligned} \tag{3.3}$$

The necessary condition for stability may then be written as

$$G_2[\underset{\sim}{u}] \geqslant 0 \tag{3.4}$$

for all $\underset{\sim}{u}$ satisfying (2.19) and the linearized incompressibility condition

$$\lambda_1^{-1}u_{1,1} + \lambda_2^{-1}u_{2,2} + E = 0 \tag{3.5}$$

which is obtained from (2.5).

For any specified value of λ_3, a critical value of λ, for which a bifurcation solution of the static problem exists, occurs when

$$\delta G_2[\underset{\sim}{u}] = 0 \tag{3.6}$$

for some non-trivial displacement field $\underset{\sim}{u}$ which satisfies the boundary conditions of the problem. We shall call the state I corresponding to such a value of λ a <u>critical state</u>. With the definition of stability given at the end of §2, state I will be stable if $G_2[\underset{\sim}{u}]$ is positive definite in a neighborhood of $\underset{\sim}{u} = \underset{\sim}{0}$. We say that a critical state is at the <u>stability limit</u> if $G_2[\underset{\sim}{u}]$ has a zero stationary value, i.e. if $G_2[\underset{\sim}{u}] = 0$ for a non-trivial value of $\underset{\sim}{u}$ satisfying (3.6) and (3.5) together with the boundary conditions of the problem.

In order to determine such a value of $\underset{\sim}{u}$, we proceed in the following manner. We take account of the constraint condition (3.5) by introducing the Lagrange multiplier $-4\ell_3 p(\xi_1,\xi_2)$ and obtain from (3.1)

$$\begin{aligned}\delta G_2[\underset{\sim}{u}] = 2\ell_3 \int\int [&k_1\{\delta\kappa[\underset{\sim}{u}] + 2(\lambda_3^2+2\lambda_2^2)E\delta E\} \\ &+ \lambda_3^2 W_2\{4\lambda_1(1-\lambda^2)(E\delta u_{1,1}+u_{1,1}\delta E) + 2\Lambda_1 E\delta E\} \\ &+ 2k_2 k\delta k + 2k_{21}\Lambda_2(E\delta k+k\delta E) \\ &+ 4\lambda_3^4 W_{22}\Lambda_2^2 E\delta E - 2p(\lambda_1^{-1}\delta u_{1,1}+\lambda_2^{-1}\delta u_{2,2}+\delta E)]d\xi_1 d\xi_2,\end{aligned} \tag{3.7}$$

where, from (2.10) and $(3.3)_1$,

$$\delta\kappa[\underset{\sim}{u}] = 2\{[(u_{1,1}-\lambda u_{2,2})\delta u_1 + (u_{2,1}+\lambda u_{1,2})\delta u_2]_{,1}$$
$$+ [(u_{2,2}-\lambda u_{1,1})\delta u_2 + (u_{1,2}+\lambda u_{2,1})\delta u_1]_{,2}$$
$$- (u_{1,11}+u_{1,22})\delta u_1 - (u_{2,11}+u_{2,22})\delta u_2\} ,$$
$$k\delta k = \lambda_1^2(1-\lambda^2)^2\{(u_{1,1}\delta u_1)_{,1} - u_{1,11}\delta u_1\}$$
$$+ \lambda_1(1-\lambda^2)(\lambda_3^2-\lambda_2^2)\{(E\delta u_1)_{,1} + u_{1,1}\delta E\} + (\lambda_3^2-\lambda_2^2)^2 E\delta E . \tag{3.8}$$

With (3.8) and $(3.3)_1$, we obtain from (3.7)

$$\delta G_2[\underset{\sim}{u}] = 4\ell_3\left\{\int_{-\ell_2}^{\ell_2} \Big[\delta u_1 F_{11}+\delta u_2 F_{21}\Big]_{-\ell_1}^{\ell_1} d\xi_2 + \int_{-\ell_1}^{\ell_1} \Big[\delta u_1 F_{12}+\delta u_1 F_{22}\Big]_{-\ell_2}^{\ell_2} d\xi_1 - \int\int (F_{1\alpha,\alpha}\delta u_1 + F_{2\alpha,\alpha}\delta u_2 - F_{33}\delta E)\,d\xi_1 d\xi_2\right\} , \tag{3.9}$$

where

$$F_{11} = \{k_1+\lambda_1^2(1-\lambda^2)^2 k_2\}u_{1,1} - \lambda k_1 u_{2,2}$$
$$+ \lambda_1(1-\lambda^2)E\{2\lambda_3^2 W_2 + (\lambda_3^2-\lambda_2^2)k_2 + \Lambda_2 k_{21}\} - \lambda_1^{-1}p ,$$
$$F_{21} = k_1(u_{2,1}+\lambda u_{1,2}) , \quad F_{12} = k_1(u_{1,2}+\lambda u_{2,1}) ,$$
$$F_{22} = k_1(u_{2,2}-\lambda u_{1,1}) - \lambda_2^{-1}p , \tag{3.10}$$
$$F_{33} = E\{k_1(\lambda_3^2+2\lambda_2^2) + \lambda_3^2\Lambda_1 W_2$$
$$+ (\lambda_3^2-\lambda_2^2)[(\lambda_3^2-\lambda_2^2)k_2 + 2\Lambda_2 k_{21}] + 2\lambda_3^4\Lambda_2^2 W_{22}\}$$
$$+ \lambda_1(1-\lambda^2)u_{1,1}\{2\lambda_3^2 W_2 + (\lambda_3^2-\lambda_2^2)k_2 + \Lambda_2 k_{21}\} - p .$$

From (2.19), δu_1 is constant on each of the faces $\xi_1 = \pm\ell_1$. Then, from (3.9) the condition $\delta G_2 = 0$ implies

$$F_{1\alpha,\alpha} = F_{2\alpha,\alpha} = 0 \ , \quad \int\int F_{33} d\xi_1 d\xi_2 = 0 \ , \tag{3.11}$$

together with the boundary conditions

$$F_{21} = 0 \ , \int_{-\ell_2}^{\ell_2} F_{11} d\xi_2 = 0 \quad \text{on} \quad \xi_1 = \pm \ell_1 \tag{3.12}$$

and

$$F_{12} = F_{22} = 0 \quad \text{on} \quad \xi_2 = \pm \ell_2 \ . \tag{3.13}$$

Solutions of the homogeneous system of equations (3.5) and (3.11)-(3.13) can be found of the form

$$\begin{gathered} u_1 = \left.\begin{matrix} -\sin\Omega\xi_1 \\ \cos\Omega\xi_1 \end{matrix}\right\} U_1(\xi_2) \ , \quad u_2 = \left.\begin{matrix} \cos\Omega\xi_1 \\ \sin\Omega\xi_1 \end{matrix}\right\} U(\xi_2) \ , \\ E = 0 \ , \quad p = \left.\begin{matrix} \cos\Omega\xi_1 \\ \sin\Omega\xi_1 \end{matrix}\right\} P(\xi_2), \end{gathered} \tag{3.14}$$

where

$$\Omega = n\pi/2\ell_1 \ , \quad n = 1,2,\ldots, \tag{3.15}$$

the upper (lower) solution corresponding to n even (odd). The functions $U_1(\xi_2)$ and $P(\xi_2)$ are given in terms of U by

$$U_1 = \frac{U'}{\lambda\Omega} \ , \quad P = \frac{\lambda_1 k_1 \beta'}{\lambda\Omega^2} \ , \tag{3.16}$$

where the prime denotes differentiation with respect to ξ_2 and

$$\beta = U'' - \{1+(\lambda-1)^2 A\}\Omega^2 U \ , \tag{3.17}$$

with

$$A = \frac{(\lambda+1)^2 k_2}{\lambda\lambda_3 k_1} \ . \tag{3.18}$$

$U(\xi_2)$ satisfies the differential equation

$$U^{(iv)} - \{\lambda^2+1+(\lambda-1)^2 A\}\Omega^2 U'' + \lambda^2\Omega^4 U = 0 \ , \tag{3.19}$$

and the boundary conditions

$$\left.\begin{aligned} &U'' + \lambda^2\Omega^2 U = 0 \\ &U''' - \{2\lambda^2+1+(\lambda-1)^2 A\}\Omega^2 U' = 0 \end{aligned}\right\} \quad \text{on } \xi_2 = \pm \ell_2 \ . \tag{3.20}$$

Solutions of (3.19) with (3.20) have already been obtained (see, for example, [1,2]). These are either even or odd functions of ξ_2 and represent respectively flexural or barreling deformations.

From (3.1) we obtain, with (2.10), $(3.3)_1$, (3.10) and $(3.14)_3$,

$$\begin{aligned} G_2[\underset{\sim}{u}] &= 2\ell_3 \int\int \{k_1\kappa[\underset{\sim}{u}] + \lambda_1^2(1-\lambda^2)^2 k_2 u_{1,1}^2\} d\xi_1 d\xi_2 \\ &= 2\ell_3 \int\int \{(u_\alpha F_{\alpha\beta})_{,\beta} - u_\alpha F_{\alpha\beta,\beta} + p(\lambda_1^{-1}u_{1,1} + \lambda_2^{-1}u_{2,2})\} d\xi_1 d\xi_2 \ . \end{aligned} \tag{3.21}$$

We have, with (3.12) and (3.13),

$$\int\int (u_\alpha F_{\alpha\beta})_{,\beta} d\xi_1 d\xi_2 = \int_{-\ell_1}^{\ell_1} [u_\alpha F_{\alpha 2}]_{-\ell_2}^{\ell_2} d\xi_1 + \int_{-\ell_2}^{\ell_2} [u_\alpha F_{\alpha 1}]_{-\ell_1}^{\ell_1} d\xi_2 = 0 \ . \tag{3.22}$$

Then, with $(3.11)_{1,2}$, (3.5), $(3.14)_3$ and (3.22), equation (3.21) yields

$$G_2[\underset{\sim}{u}] = 0 \, . \tag{3.23}$$

Accordingly, for any displacement field of the form (3.14) the critical states are states of neutral stability.

Substituting from (3.14) in $(2.10)_2$, we obtain, with $(3.16)_1$,

$$\begin{aligned}\kappa[\underset{\sim}{u}] = (2\lambda^2\Omega^2)^{-1}[&(\lambda^2+1)\Omega^2U'^2 + U''^2 + \lambda^2\Omega^4U^2 + 2\lambda^2\Omega^2(UU')' \\ &+ (-1)^n\cos 2\Omega\xi_1\{(\lambda^2+1)\Omega^2U'^2 - U''^2 - \lambda^2\Omega^4U^2 - 2\lambda^2\Omega^2\alpha\}],\end{aligned} \tag{3.24}$$

where

$$\alpha = UU'' - U'^2 \, . \tag{3.25}$$

We note that $\kappa[\underset{\sim}{u}]$ is necessarily an even function of ξ_2 .

4. Potential energy near a critical state

Paralleling the procedure of our previous paper [3], we develop the stability condition that $G[\underset{\sim}{u}]$ be positive definite for all $\underset{\sim}{u}$ satisfying the kinematic constraints (2.5) and (2.19) and lying in a neighborhood of $\underset{\sim}{u} = \underset{\sim}{0}$.

We take

$$\underset{\sim}{u} = \varepsilon\hat{\underset{\sim}{u}} + \varepsilon^2\bar{\underset{\sim}{u}} , \tag{4.1}$$

where $\hat{\underset{\sim}{u}}$ is the solution for $\underset{\sim}{u}$ given by equations (3.14)-(3.20), ε is a small parameter, and $\bar{\underset{\sim}{u}}$ satisfies the orthogonality condition

$$\int\int \hat{u}_{i,m}\bar{u}_{i,m}d\xi_1 d\xi_2 = 0 . \tag{4.2}$$

We then determine the value of $\bar{\underset{\sim}{u}}$ which, for a fixed value of ε , gives a stationary value of $G[\underset{\sim}{u}]$. If this value of $G[\underset{\sim}{u}]$ is positive (negative) then the critical state is stable (unstable).

With $(3.14)_3$ and (2.4), we can rewrite (4.1) as

$$u_\alpha = \varepsilon\hat{u}_\alpha + \varepsilon^2\bar{u}_\alpha , \quad E = \varepsilon^2\bar{E} , \text{ say,} \tag{4.3}$$

where, from (4.2),

$$\int\int \hat{u}_{\alpha,\beta}\bar{u}_{\alpha,\beta}d\xi_1 d\xi_2 = 0 . \tag{4.4}$$

Also, since $\underset{\sim}{u}$ must satisfy the incompressibility constraint (2.5), we obtain, with (4.3) and (3.5) and the neglect of terms of higher degree than the first in ε ,

$$\lambda_1^{-1}\bar{u}_{1,1}+\lambda_2^{-1}\bar{u}_{2,2} = -\bar{E} + \lambda_3(\hat{u}_{1,2}\hat{u}_{2,1}-\hat{u}_{1,1}\hat{u}_{2,2})$$
$$+ \varepsilon\lambda_3(\hat{u}_{1,2}\bar{u}_{2,1}+\hat{u}_{2,1}\bar{u}_{1,2}-\hat{u}_{1,1}\bar{u}_{2,2}-\hat{u}_{2,2}\bar{u}_{1,1}). \qquad (4.5)$$

Again, since $\underset{\sim}{u}$ must satisfy the constraint (2.19) on $\xi_1 = \pm \ell_1$, we have

$$\bar{u}_1(\ell_1,\xi_2) = -\bar{u}_1(-\ell_1,\xi_2) = \lambda_1\bar{e}\ell_1, \qquad (4.6)$$

where $\bar{e}$ is a constant.

It is shown in Appendix A that, with (4.3), $G[\underset{\sim}{u}]$, given by (2.22), may be expressed to order ε^4 in the form (A8), thus

$$G[\underset{\sim}{u}] = G[\varepsilon\hat{\underset{\sim}{u}}+\varepsilon^2\bar{\underset{\sim}{u}}]$$
$$= \varepsilon^2G_2[\hat{\underset{\sim}{u}}] + 2\ell_3\varepsilon^3 \int\int (g_1^{(3)}+g_2^{(3)}+\varepsilon g^{(4)})d\xi_1 d\xi_2 , \qquad (4.7)$$

where the g's are defined in (A9). From (3.23), $G_2[\hat{\underset{\sim}{u}}] = 0$. We will now show that the remaining terms on the right-hand side of (4.7) are of order ε^4 .

We replace u_α,p,E by $\hat{u}_\alpha,\hat{p},\hat{E} = 0$ in (3.10) and use the resulting equations, together with (A3) and $(A9)_1$, to express $g_1^{(3)}$ in the form

$$g_1^{(3)} = 2\{(\bar{u}_\alpha F_{\alpha\beta})_{,\beta}-\bar{u}_\alpha F_{\alpha\beta,\beta}+\hat{p}(\lambda_1^{-1}\bar{u}_{1,1}+\lambda_2^{-1}\bar{u}_{2,2})\} . \qquad (4.8)$$

From $(3.14)_{1,2}$, $(3.16)_1$ and (3.25) we obtain

$$\hat{u}_{1,2}\hat{u}_{2,1} - \hat{u}_{1,1}\hat{u}_{2,2} = \frac{1}{2\lambda}\{(UU')' - (-1)^n \alpha\cos 2\Omega\xi_1\} . \tag{4.9}$$

Also, from $(3.14)_{1,4}$, (3.15), (3.24) and (4.9), we obtain

$$\int_{-\ell_1}^{\ell_1} \hat{u}_{1,1}d\xi_1 = \int_{-\ell_1}^{\ell_1} \hat{u}_{1,1}\kappa[\hat{\underset{\sim}{u}}]d\xi_1 = \int_{-\ell_1}^{\ell_1} (\hat{u}_{1,1})^3 d\xi_1$$

$$= \int_{-\ell_1}^{\ell_1} \hat{p}d\xi_1 = \int_{-\ell_1}^{\ell_1} \hat{p}(\hat{u}_{1,2}\hat{u}_{2,1}-\hat{u}_{1,1}\hat{u}_{2,2})d\xi_1 = 0 . \tag{4.10}$$

We now substitute from (4.5) in (4.8). From the resulting expression for $g_1^{(3)}$ and (4.10), (3.11)-(3.13), we obtain, to order ε ,

$$\int\int g_1^{(3)}d\xi_1 d\xi_2 = 2\varepsilon\lambda_3 \int\int \hat{p}(\hat{u}_{1,2}\overline{u}_{2,1}+\hat{u}_{2,1}\overline{u}_{1,2}-\hat{u}_{1,1}\overline{u}_{2,2}$$

$$-\hat{u}_{2,2}\overline{u}_{1,1})d\xi_1 d\xi_2 . \tag{4.11}$$

Also, from the expression $(A9)_2$ for $g_2^{(3)}$ and (4.10), we obtain

$$\int\int g_2^{(3)}d\xi_1 d\xi_2 = 0 . \tag{4.12}$$

With (4.11), (4.12) and $G_2[\hat{\underset{\sim}{u}}] = 0$, equation (4.7) yields

$$G[\underset{\sim}{u}] = \varepsilon^4\overline{G}[\overline{\underset{\sim}{u}}] = 2\ell_3\varepsilon^4 \int\int \overline{g}d\xi_1 d\xi_2 , \tag{4.13}$$

where

$$\overline{g} = g^{(4)} + 2\lambda_3\hat{p}(\hat{u}_{1,2}\overline{u}_{2,1}+\hat{u}_{2,1}\overline{u}_{1,2}-\hat{u}_{1,1}\overline{u}_{2,2}-\hat{u}_{2,2}\overline{u}_{1,1}) . \tag{4.14}$$

5. Conditions for $\bar{G}$ to have a stationary value

In this section we shall obtain differential equations and boundary conditions determining the field $\underset{\sim}{\bar{u}}$ which satisfies the kinematic constraint (4.5) and the orthogonality constraint (4.4) and leads to a stationary value for $\bar{G}$. We use the method of undetermined multipliers to remove the constraints and accordingly write, neglecting terms of order ε in (4.5),

$$\bar{G}[\underset{\sim}{\bar{u}}] = 2\ell_3 \int\int \{\bar{g} - 2\bar{p}[\lambda_1^{-1}\bar{u}_{1,1} + \lambda_2^{-1}\bar{u}_{2,2} + \bar{E} - \lambda_3(\hat{u}_{1,2}\hat{u}_{2,1} - \hat{u}_{1,1}\hat{u}_{2,2})] + 2\bar{\chi}\hat{u}_{\alpha,\beta}\bar{u}_{\alpha,\beta}\} d\xi_1 d\xi_2 \,, \tag{5.1}$$

where $-4\ell_3\bar{p} = -4\ell_3\bar{p}(\xi_1,\xi_2)$ and $4\ell_3\bar{\chi}$ = constant are the Lagrange multipliers associated with the constraints (4.5) and (4.4) respectively. Then for $\bar{G}[\underset{\sim}{\bar{u}}]$ to have a stationary value,

$$\delta\bar{G}[\underset{\sim}{\bar{u}}] = 0 \tag{5.2}$$

for all $\underset{\sim}{\bar{u}}$ in the neighborhood of $\underset{\sim}{\bar{u}} = \underset{\sim}{0}$.

From (5.1) we obtain, with (4.14), $(A9)_4$, (A1) and (A3),

$$\begin{aligned}\delta\bar{G}[\underset{\sim}{\bar{u}}] &= 4\ell_3 \int\int (\bar{F}_{\alpha\beta}\delta\bar{u}_{\alpha,\beta} + \bar{F}_{33}\delta\bar{E}) d\xi_1 d\xi_2 \\ &= 4\ell_3\Big\{\int_{-\ell_2}^{\ell_2} \Big[\bar{F}_{\alpha 1}\delta\bar{u}_\alpha\Big]_{-\ell_1}^{\ell_1} d\xi_2 + \int_{-\ell_1}^{\ell_1} \Big[\bar{F}_{\alpha 2}\delta\bar{u}_\alpha\Big]_{-\ell_2}^{\ell_2} d\xi_1 \\ &\quad - \int\int (\bar{F}_{\alpha\beta,\beta}\delta\bar{u}_\alpha - \bar{F}_{33}\delta\bar{E})\delta\xi_1\delta\xi_2\Big\} ,\end{aligned} \tag{5.3}$$

where $\bar{F}_{\alpha\beta}$ and $\bar{F}_{33}$ are defined by

$$
\begin{aligned}
\overline{F}_{11} = {} & \{k_1+\lambda_1^2(\lambda^2-1)^2k_2\}\overline{u}_{1,1}-\lambda k_1\overline{u}_{2,2}-\lambda_1^{-1}\overline{p}-\lambda_3\hat{p}\hat{u}_{2,2} \\
& -\lambda_1(\lambda^2-1)\{2\lambda_3^2W_2-(\lambda_2^2-\lambda_3^2)k_2+\Lambda_2k_{21}\}\overline{E} \\
& -\lambda_1(\lambda^2-1)k_2\{\tfrac{1}{2}\kappa[\hat{\underset{\sim}{u}}] + \hat{u}_{1,1}(\hat{u}_{1,1}-\lambda\hat{u}_{2,2})\} \\
& -12\lambda_1^3(\lambda^2-1)^3k_3(\hat{u}_{1,1})^2 + \overline{\chi}\hat{u}_{1,1} \;, \\
\overline{F}_{21} = {} & k_1(\overline{u}_{2,1}+\lambda\overline{u}_{1,2}) + \lambda_3\hat{p}\hat{u}_{1,2} \\
& -\lambda_1(\lambda^2-1)k_2\hat{u}_{1,1}(\hat{u}_{2,1}+\lambda\hat{u}_{1,2}) + \overline{\chi}\hat{u}_{2,1} \;, \\
\overline{F}_{12} = {} & k_1(\overline{u}_{1,2}+\lambda\overline{u}_{2,1}) + \lambda_3\hat{p}\hat{u}_{2,1} \\
& -\lambda_1(\lambda^2-1)k_2\hat{u}_{1,1}(\hat{u}_{1,2}+\lambda\hat{u}_{2,1}) + \overline{\chi}\hat{u}_{1,2} \;, \qquad (5.4) \\
\overline{F}_{22} = {} & k_1(\overline{u}_{2,2}-\lambda\overline{u}_{1,1}) - \lambda_2^{-1}\overline{p}-\lambda_3\hat{p}\hat{u}_{1,1} \\
& -\lambda_1(\lambda^2-1)k_2\hat{u}_{1,1}(\hat{u}_{2,2}-\lambda\hat{u}_{1,1}) + \overline{\chi}\hat{u}_{2,2} \;, \\
\overline{F}_{33} = {} & \{(\lambda_3^2+2\lambda_2^2)k_1+\lambda_3^2\Lambda_1W_2+(\lambda_2^2-\lambda_3^2)^2k_2 \\
& -2(\lambda_2^2-\lambda_3^2)\Lambda_2k_{21}+2\lambda_3^4\Lambda_2^2W_{22}\}\overline{E} - \overline{p} \\
& -\lambda_1(\lambda^2-1)\{2\lambda_3^2W_2-(\lambda_2^2-\lambda_3^2)k_2+\Lambda_2k_{21}\}\overline{u}_{1,1} \\
& +\{\lambda_3^2W_2-\tfrac{1}{2}(\lambda_2^2-\lambda_3^2)k_2+\tfrac{1}{2}\Lambda_2k_{21}\}\kappa[\hat{\underset{\sim}{u}}] \\
& +2\lambda_1^2(\lambda^2-1)^2\{k_{21}-6(\lambda_2^2-\lambda_3^2)k_3+2\Lambda_2k_{31}\}(\hat{u}_{1,1})^2 .
\end{aligned}
$$

With (5.3) and (4.6) the condition (5.2) yields

$$
\overline{F}_{1\alpha,\alpha} = \overline{F}_{2\alpha,\alpha} = \int\!\!\int \overline{F}_{33}d\xi_1d\xi_2 = 0 \qquad (5.5)
$$

and

$$
\begin{aligned}
\overline{F}_{21} &= \int_{-\ell_2}^{\ell_2} \overline{F}_{11}d\xi_2 = 0 \quad \text{on} \quad \xi_1 = \pm\ell_1 \;, \\
\overline{F}_{12} &= \overline{F}_{22} = 0 \qquad\qquad\; \text{on} \quad \xi_2 = \pm\ell_2 \;.
\end{aligned} \qquad (5.6)
$$

These relations have the same form as (3.11)-(3.13) with $F_{\alpha\beta}$ replaced by $\bar{F}_{\alpha\beta}$.

Neglecting terms of order ε in the constraint (4.5), we obtain

$$\lambda_1^{-1}\bar{u}_{1,1} + \lambda_2^{-1}\bar{u}_{2,2} = -\bar{E} + \lambda_3(\hat{u}_{1,2}\hat{u}_{2,1} - \hat{u}_{1,1}\hat{u}_{2,2}) \ . \tag{5.7}$$

With the expressions (5.4) for $\bar{F}_{\alpha\beta}$ and $\bar{F}_{33}$ and the expressions for $\hat{\underset{\sim}{u}}$ and $\hat{p}$ obtained in §3, equations (5.5)-(5.7), together with the orthogonality condition (4.4) and the additional boundary condition (4.6), provide a set of equations for the determination of $\bar{u}_\alpha$, $\bar{E}$, $\bar{e}$, $\bar{p}$ and $\bar{\chi}$.

6. Development of the governing equations for $\underset{\sim}{\bar{u}}$

In this section we obtain from (5.4)-(5.7) expressions for $\underset{\sim}{\bar{u}}$ analogous to those given for $\underset{\sim}{\hat{u}}$ in (3.14).

We first use the expressions obtained in §3 to substitute for $\hat{u}_\alpha$ and $\hat{p}$ in the expressions (5.4) for $\bar{F}_{\alpha\beta}$ and $\bar{F}_{33}$ and obtain the new expressions for $\bar{F}_{\alpha\beta}$ and $\bar{F}_{33}$ given by equations (B1)-(B3) of Appendix B.

From (3.14), (3.15) and (4.6), we obtain

$$\hat{u}_{1,2} = \hat{u}_{2,1} = \bar{u}_{1,2} = 0 \quad \text{on} \quad \xi_1 = \pm \ell_1 . \tag{6.1}$$

Accordingly, with the expressions $(5.4)_2$ for $\bar{F}_{21}$, the boundary condition $(5.6)_1$ may be replaced by

$$\bar{u}_{2,1} = 0 \quad \text{on} \quad \xi_1 = \pm \ell_1 . \tag{6.2}$$

From (5.7) and (4.9) we obtain

$$\lambda\bar{u}_{1,1}+\bar{u}_{2,2}+\lambda_2\bar{E} = (2\lambda_2)^{-1}\{(UU')'-(-1)^n\alpha\cos 2\Omega\xi_1\} . \tag{6.3}$$

We shall assume a solution for $\bar{u}_\alpha$, $\bar{E}$, $\bar{e}$, $\bar{p}$ and $\bar{\chi}$ of the equations (5.5)-(5.7), (4.4) and (4.6) of the form

$$\begin{aligned}
\bar{u}_1 &= \bar{U}_1 \sin 2\Omega\xi_1 + \lambda_1\bar{e}\xi_1 , \\
\bar{u}_2 &= \bar{U}\cos 2\Omega\xi_1 + \frac{1}{2\lambda_2} UU' - \lambda_2(\bar{e}+\bar{E})\xi_2 , \\
\bar{p} &= \bar{P}\cos 2\Omega\xi_1 + Q , \\
\bar{\chi} &= 0 ,
\end{aligned} \tag{6.4}$$

where $\bar{U}$, $\bar{U}_1$, $\bar{P}$ and Q are functions of ξ_2 only, and U is a function of ξ_2 only determined by equations (3.19) and (3.20).

We note from (3.14) and (3.15) that expressions for $\bar{u}_\alpha$ of the form $(6.4)_{1,2}$ automatically satisfy the orthogonality condition (4.4), the boundary condition (4.6), and the boundary condition (6.2) which is equivalent to $(5.6)_1$. By substituting from $(6.4)_{1,2}$ in (6.3) it follows that

$$\bar{U}_1 = -\frac{1}{2\lambda\Omega}\{\bar{U}' + \frac{(-1)^n}{2\lambda_2}\alpha\} , \tag{6.5}$$

where α is defined in (3.25).

Introducing the expressions (B1) for $\bar{F}_{\alpha\beta}$ given in Appendix B into $(5.5)_{1,2}$, we obtain with $(6.4)_4$ and (3.18) the differential equations

$$\begin{aligned} k_1\{[1+(\lambda-1)^2A]\bar{u}_{1,11}+\bar{u}_{1,22}\}-\lambda_1^{-1}\bar{p}_{,1} &= (-1)^n(\phi_{12}'-2\Omega\phi_{11})\sin 2\Omega\xi_1 , \\ k_1(\bar{u}_{2,11}+\bar{u}_{2,22})-\lambda_2^{-1}\bar{p}_{,2} &= f_{22}'+(-1)^n(\phi_{22}'+2\Omega\phi_{21})\cos 2\Omega\xi_1 , \end{aligned} \tag{6.6}$$

where f_{22} and $\phi_{\alpha\beta}$ are defined in equations (B2) and (B3) of Appendix B. Now, substituting in (6.6) for $\bar{u}_\alpha$ and $\bar{p}$ from $(6.4)_{1,2,3}$ we obtain

$$\begin{aligned} &k_1\{\bar{U}_1''-4\Omega^2[1+(\lambda-1)^2A]\bar{U}_1\} + 2\Omega\lambda_1^{-1}\bar{P} = (-1)^n(\phi_{12}'-2\Omega\phi_{11}) , \\ &k_1(\bar{U}''-4\Omega^2\bar{U}) -\lambda_2^{-1}\bar{P}' = (-1)^n(\phi_{22}'+2\Omega\phi_{21}) , \\ &k_1(UU')'' - 2Q' = 2\lambda_2 f_{22}' . \end{aligned} \tag{6.7}$$

Eliminating $\bar{P}$ from $(6.7)_{1,2}$ and using (6.5), we obtain

$$\bar{U}^{(iv)}-4\Omega^2\{\lambda^2+1+(\lambda-1)^2A\}\bar{U}'' + 16\lambda^2\Omega^4\bar{U} = -(-1)^n\lambda_3^{\frac{1}{2}}\phi' , \tag{6.8}$$

where

$$\Phi' = \lambda_3^{-\frac{1}{2}}\{q' + \frac{4\lambda^2\Omega^2}{k_1}(\phi'_{22}+2\Omega\phi_{21})\} ,$$

and (6.9)

$$q = \frac{1}{2\lambda_2}\{\alpha''-4\Omega^2\alpha[1+(\lambda-1)^2A] + \frac{4\lambda\lambda_2\Omega}{k_1}(\phi'_{12}-2\Omega\phi_{11})\} .$$

It is shown in Appendix B that Φ may be expressed in the form given in equation (B11).

With (6.5), equations $(6.7)_{1,3}$ yield

$$\bar{P} = \frac{\lambda_1 k_1}{4\lambda\Omega^2}\{\bar{U}''' - 4\Omega^2[1+(\lambda-1)^2A]\bar{U}' + (-1)^n q\} ,$$

$$Q' = \tfrac{1}{2}k_1(UU')'' - \lambda_2 f'_{22} . \qquad (6.10)$$

With equations (B1) in Appendix B and $(6.4)_4$, the boundary conditions $(5.6)_{3,4}$ yield

$$\left.\begin{aligned} k_1(\bar{u}_{1,2}+\lambda\bar{u}_{2,1}) &= (-1)^n\phi_{12}\sin 2\Omega\xi_1 , \\ k_1(\bar{u}_{2,2}-\lambda\bar{u}_{1,1})-\lambda_2^{-1}\bar{p} &= f_{22}+(-1)^n\phi_{22}\cos 2\Omega\xi_1 \end{aligned}\right\} \text{ on } \xi_2 = \pm \ell_2 . \qquad (6.11)$$

Then, substituting from (6.4) in (6.11), we obtain, with (6.5) and $(6.10)_1$,

$$\left.\begin{aligned} \bar{U}'' + 4\lambda^2\Omega^2\bar{U} &= -(-1)^n\lambda_3^{\frac{1}{2}}\phi_1 , \\ \bar{U}''' - 4\Omega^2\{2\lambda^2 + (\lambda-1)^2A\}\bar{U}' &= (-1)^n\lambda_3^{\frac{1}{2}}\phi_2 \end{aligned}\right\} \text{ on } \xi_2 = \pm \ell_2 , \qquad (6.12)$$

where

$$\Phi_1 = \frac{1}{2\lambda_2\lambda_3^{\frac{1}{2}}} (\alpha' + \frac{4\lambda\lambda_2\Omega}{k_1} \phi_{12}) ,$$
$$\Phi_2 = -\lambda_3^{-\frac{1}{2}}\{q - \frac{2\lambda^2\Omega^2}{\lambda_2} (\alpha - \frac{2\lambda_2}{k_1} \phi_{22})\} . \tag{6.13}$$

It is shown in Appendix B that Φ_1 and Φ_2 may be expressed in the forms given in equations (B14).

We also obtain on $\xi_2 = \pm \ell_2$,

$$Q = k_1[\tfrac{1}{2}(UU')' - \lambda_2^2(2\bar{e} + \bar{E})] - \lambda_2 f_{22} . \tag{6.14}$$

It follows, with $(6.10)_2$, that (6.14) is valid throughout the body.

The relations $(5.6)_2$ and $(5.5)_3$ express the assumptions of dead-loading in the 1 and 3 directions respectively. We substitute in them the expressions for $\bar{F}_{11}$ and $\bar{F}_{33}$ given in equations (B1) of Appendix B. Then, using (3.2), (3.3), (3.15), (3.18), (6.4), (6.5), (6.9), (6.10), and $(6.14)_1$, we obtain

$$a_1\bar{e} + a\bar{E} = \lambda_3 b_1 , \quad a\bar{e} + a_2\bar{E} = \lambda_3 b_2 , \tag{6.15}$$

where

$$a_1 = 3\lambda^2 + 1 + (\lambda-1)^2 A ,$$
$$a = \frac{2}{k_1\lambda_1^2} \{\lambda_2^2 W_1 + \lambda_1^2\lambda_3^2 W_2 + (\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)[W_{11}$$
$$+ (\lambda_1^2 + \lambda_3^2) W_{12} + \lambda_1^2\lambda_3^2 W_{22}]\} , \tag{6.16}$$

$$a_2 = \frac{1}{k_1\lambda_1^2}\{(3\lambda_2^2+\lambda_3^2)(W_1+\lambda_1^2W_2) + 2(\lambda_2^2-\lambda_3^2)^2(W_{11}+2\lambda_1^2W_{12}+\lambda_1^4W_{22})\} ,$$

and

$$b_1 = \frac{(-1)^n\lambda_1}{8\ell_2\Omega^2}\int_{-\ell_2}^{\ell_2}(\bar{U}''+4\lambda^2\Omega^2\bar{U})'d\xi_2 + \frac{\lambda}{2\ell_2}\int_{-\ell_2}^{\ell_2}\{(UU')' + \frac{\alpha}{2\lambda^2}[1+(\lambda-1)^2A] + \frac{\lambda_1 q}{4\lambda\Omega^2} + \frac{\lambda_1}{k_1}(f_{11}+\phi_{11}-\lambda f_{22})\}d\xi_2 ,$$

$$b_2 = \frac{\lambda}{4\ell_2}\int_{-\ell_2}^{\ell_2}\{(UU')' + \frac{2}{k_1}(f_{33}-\lambda_2 f_{22})\}d\xi_2 . \tag{6.17}$$

With the relations $(6.12)_1$, $(6.13)_1$ and $(6.9)_2$, equation $(6.17)_1$ yields

$$b_1 = \frac{\lambda}{2\ell_2}\int_{-\ell_2}^{\ell_2}\{(UU')' + \frac{\lambda_1}{k_1}(f_{11}-\lambda f_{22})\}d\xi_2 . \tag{6.18}$$

The expressions for b_1 and b_2 may be rewritten in the forms given in equations (B16) of Appendix B.

Equations (6.15) may be solved for $\bar{e}$ and $\bar{E}$ to yield

$$\bar{e} = \frac{\lambda_3}{\Delta}(b_1a_2-b_2a) , \quad \bar{E} = \frac{\lambda_3}{\Delta}(b_2a_1-b_1a) , \tag{6.19}$$

where Δ , assumed to be non-zero, is defined by

$$\Delta = a_1a_2-a^2 . \tag{6.20}$$

From (6.19), we obtain

$$b_1\bar{e} + b_2\bar{E} = \frac{\lambda_3}{\Delta}(a_2b_1^2+a_1b_2^2-2ab_1b_2) . \tag{6.21}$$

7. Development of the expression for $\bar{G}[\bar{\underset{\sim}{u}}]$

In this section the expressions obtained for $\hat{\underset{\sim}{u}}$ and $\hat{p}$ in §3 and the expressions obtained for $\bar{\underset{\sim}{u}}$ in §6 are used to write the expression for $\bar{G}[\bar{\underset{\sim}{u}}]$ given by (4.13) and (4.14) in terms of the functions $U(\xi_2)$ and $\bar{U}(\xi_2)$ only. With $(2.10)_2$, (5.4), $(5.6)_1$ and equations (A1), (A3) and $(A9)_3$ in Appendix A, we can rewrite the expression (4.14) for $\bar{g}$ as

$$\begin{aligned}\bar{g} = {} & (\bar{u}_\alpha \bar{F}_{\alpha\beta})_{,\beta} - \bar{u}_\alpha \bar{F}_{\alpha\beta,\beta} + \bar{u}_{\alpha,\beta} H_{\alpha\beta} + \bar{E}(\bar{F}_{33} + H_{33}) + H \\ & + \lambda_3 \bar{p}(\hat{u}_{1,2}\hat{u}_{2,1} - \hat{u}_{1,1}\hat{u}_{2,2}) \ ,\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
H_{11} &= -\lambda_3 \hat{p}\hat{u}_{2,2} - \lambda_1(\lambda^2-1)k_2\{\tfrac{1}{2}\kappa[\hat{\underset{\sim}{u}}] + \hat{u}_{1,1}(\hat{u}_{1,1} - \lambda\hat{u}_{2,2})\} \\
&\qquad - 12\lambda_1^3(\lambda^2-1)^3 k_3(\hat{u}_{1,1})^2 \ , \\
H_{22} &= -\lambda_3 \hat{p}\hat{u}_{1,1} - \lambda_1(\lambda^2-1)k_2\hat{u}_{1,1}(\hat{u}_{2,2} - \lambda\hat{u}_{1,1}) \ , \\
H_{12} &= \lambda_3 \hat{p}\hat{u}_{2,1} - \lambda_1(\lambda^2-1)k_2\hat{u}_{1,1}(\hat{u}_{1,2} + \lambda\hat{u}_{2,1}) \ , \\
H_{21} &= \lambda_3 \hat{p}\hat{u}_{1,2} - \lambda_1(\lambda^2-1)k_2\hat{u}_{1,1}(\hat{u}_{2,1} + \lambda\hat{u}_{1,2}) \ , \\
H_{33} &= \{\lambda_3^2 W_2 - \tfrac{1}{2}(\lambda_2^2 - \lambda_3^2)k_2 + \tfrac{1}{2}\Lambda_2 k_{21}\}\kappa[\hat{\underset{\sim}{u}}] \\
&\qquad + 2\lambda_1^2(\lambda^2-1)^2\{k_{21} - 6(\lambda_2^2 - \lambda_3^2)k_3 + 2\Lambda_2 k_{31}\}(\hat{u}_{1,1})^2 \ , \\
H &= \tfrac{1}{4}k_2(\kappa[\hat{\underset{\sim}{u}}])^2 + 12\lambda_1^2(\lambda^2-1)^2 k_3(\hat{u}_{1,1})^2\kappa[\hat{\underset{\sim}{u}}] \\
&\qquad + 16\lambda_1^4(\lambda^2-1)^4 k_4(\hat{u}_{1,1})^4 \ .
\end{aligned} \tag{7.2}$$

We now substitute in (7.2) the expressions for $\hat{u}_\alpha$ and $\hat{p}$ given in (3.14) and use (3.16), (3.18) and (3.24). Then, with the notation introduced in equations $(B4)_2$, (B5) and (B17) of

Appendix B and the further notation (cf.(3.24))

$$\kappa_1 = (\lambda^2+1)\Omega^2U'^2 + U''^2 + \lambda^2\Omega^4U^2 + 2\lambda^2\Omega^2(UU')' ,$$
$$\kappa_2 = (\lambda^2+1)\Omega^2U'^2 - U''^2 - \lambda^2\Omega^4U^2 - 2\lambda^2\Omega^2\alpha , \tag{7.3}$$

the following expressions for $H_{\alpha\beta}$ are obtained:

$$H_{11} = -\frac{k_1}{4\lambda\lambda_2\Omega^2}\{h_{11}+(-1)^n\eta_{11}\cos 2\Omega\xi_1\} ,$$
$$H_{22} = \frac{k_1}{2\lambda^2\lambda_2\Omega^2}\{h_{22}+(-1)^n\eta_{22}\cos 2\Omega\xi_1\} ,$$
$$H_{33} = \frac{k_1}{2\lambda^2\Omega^2}\{h_{33}+(-1)^n\eta_{33}\cos 2\Omega\xi_1\} , \tag{7.4}$$
$$H_{12} = -\frac{k_1}{2\lambda\lambda_2\Omega}(-1)^n\eta_{12}\sin 2\Omega\xi_1 ,$$
$$H_{21} = -\frac{k_1}{2\lambda^2\lambda_2\Omega^3}(-1)^n\eta_{21}\sin 2\Omega\xi_1 ,$$

where the h's and η's are functions of ξ_2 only given by

$$h_{11} = 2\beta'U' + \bar{A}\{\kappa_1+2(\lambda^2+1)\Omega^2U'^2\} + 4(\lambda-1)^3B\Omega^2U'^2 ,$$
$$h_{22} = \beta'U' + 2\lambda^2\bar{A}\Omega^2U'^2 , \tag{7.5}$$
$$h_{33} = \bar{c}\kappa_1 + (\lambda^2-1)^2\,\bar{d}\Omega^2U'^2 ,$$

and

$$\eta_{11} = 2\beta'U' + \bar{A}\{\kappa_2+2(\lambda^2+1)\Omega^2U'^2\} + 4(\lambda-1)^3B\Omega^2U'^2 ,$$
$$\eta_{22} = h_{22} ,$$
$$\eta_{33} = \bar{c}\kappa_2 + (\lambda^2-1)^2\bar{d}\Omega^2U'^2 , \tag{7.6}$$

$$\eta_{12} = \beta' U + \bar{A} U'(U'' + \lambda^2 \Omega^2 U) ,$$

$$\eta_{21} = \beta' U'' + \lambda^2 \bar{A} \Omega^2 U'(U'' + \Omega^2 U) .$$

We also obtain

$$H = \frac{k_1}{8\lambda^4 \Omega^4} \{\lambda\lambda_3 h + (-1)^n h_1 \cos 2\Omega\xi_1 + h_2 \cos 4\Omega\xi_1\} , \tag{7.7}$$

where

$$h = \frac{\bar{A}}{4(\lambda^2 - 1)} (2\kappa_1^2 + \kappa_2^2) + \frac{2(\lambda-1)^2 B}{\lambda+1} \Omega^2 U'^2 (2\kappa_1 + \kappa_2) + (\lambda-1)^4 C \Omega^4 U'^4 , \tag{7.8}$$

and

$$C = \frac{48(\lambda+1)^4 k_4}{\lambda^3 \lambda_3^3 k_1} .$$

The corresponding expressions for h_1 and h_2 will not be required.

We now substitute from (7.1) in (4.13) and use (4.4), (5.5) and (5.6) to obtain

$$\bar{G}[\underset{\sim}{\bar{u}}] = 2\ell_3 \int\int \{\bar{u}_{\alpha,\beta} H_{\alpha\beta} + \bar{E} H_{33} + H + \lambda_3 \bar{p}(\hat{u}_{1,2}\hat{u}_{2,1} - \hat{u}_{1,1}\hat{u}_{2,2})\} d\xi_1 d\xi_2 . \tag{7.9}$$

Then, introducing into (7.9) the expressions (7.4) and (7.7) for $H_{\alpha\beta}$ and H and the expressions for $\bar{u}_\alpha$ and $\bar{p}$ given by (6.4) and (6.5), using (4.9), and carrying out the integration with respect to ξ_1, we obtain

$$\bar{G}[\underset{\sim}{\bar{u}}] = \frac{4\ell_1\ell_3k_1}{\lambda^2\Omega^2}\left[\frac{1}{16\lambda_2^2}\int_{-\ell_2}^{\ell_2}\{(-1)^n\lambda_2 g_1+g_2\}d\xi_2 - \frac{1}{4}\int_{-\ell_2}^{\ell_2}(\bar{e}g_3+\bar{E}g_4)d\xi_2 + \frac{\lambda_3}{8\lambda\Omega^2}\int_{-\ell_2}^{\ell_2} hd\xi_2\right], \tag{7.10}$$

where

$$\begin{aligned} g_1 &= -\alpha\bar{U}''' + 2\eta_{12}\bar{U}'' + \{4\Omega^2\alpha+4(\lambda^2-1)\bar{A}\Omega^2\alpha + 2\eta_{11} \\ &\qquad + 4\eta_{22}\}\bar{U}' + 8\eta_{21}\bar{U}, \\ g_2 &= \eta_{11}\alpha + \eta_{12}\alpha' + 4h_{22}(UU')' - \lambda_2\alpha q \\ &\qquad + 4\lambda^2\Omega^2\{(UU')'\}^2 - \frac{8\lambda^2\lambda_2\Omega^2 f_{22}}{k_1}(UU')', \\ g_3 &= h_{11} + 2h_{22} + 4\lambda^2\Omega^2(UU')', \\ g_4 &= 2\{h_{22} + \lambda^2\Omega^2(UU')' - h_{33}\}. \end{aligned} \tag{7.11}$$

Introducing into $(7.11)_{3,4}$ the expressions (7.5) for h_{11}, h_{22}, h_{33}, we obtain

$$\begin{aligned} g_3 &= 4\beta'U' + 4\lambda^2\Omega^2(UU')' + 2(3\lambda^2+1)\bar{A}\Omega^2U'^2 + \bar{A}\kappa_1 \\ &\qquad + 4(\lambda-1)^3B\Omega^2U'^2, \\ g_4 &= 2\{\beta'U' + \lambda^2\Omega^2(UU')' - \bar{c}\kappa_1 + 2\lambda^2\bar{A}\Omega^2U'^2 \\ &\qquad - (\lambda^2-1)^2\bar{d}\Omega^2U'^2\}. \end{aligned} \tag{7.12}$$

From the expressions $(B7)_5$ and (7.3) for β' and κ_1 we obtain, with $(B15)_1$,

$$\beta' U' = (U'U'')' - U''^2 - \{1+(\lambda^2-1)\bar{A}\}\Omega^2 U'^2 ,$$

$$\kappa_1 = [U'(U''+\lambda^2\Omega^2 U) - U\{U''' - [2\lambda^2+1+(\lambda^2-1)\bar{A}]\Omega^2 U'\}]' - (\lambda^2-1)\bar{A}\Omega^2 U'^2 . \tag{7.13}$$

We now substitute from (7.13) in (7.12), use (B15), and integrate the resulting expressions for g_3 and g_4. Then with relations (3.20) we obtain

$$\frac{1}{4}\int_{-\ell_2}^{\ell_2} (\bar{e}g_3+\bar{E}g_4)d\xi_2 = 2\ell_2\lambda\Omega^2(b_1\bar{e}+b_2\bar{E}) , \tag{7.14}$$

where b_1 and b_2 are given by (B16).

Noting from $(7.6)_5$, with $(B7)_5$, that η_{21} may be written in the form

$$\eta_{21} = \frac{1}{2}\bar{\eta}'_{21}, \tag{7.15}$$

where

$$\bar{\eta}_{21} = U''^2+(\bar{A}-1)\Omega^2 U'^2+\bar{A}\lambda^2\Omega^4 U^2 , \tag{7.16}$$

and that

$$\alpha\bar{U}''' = (\alpha\bar{U}'')' - (\alpha'\bar{U}')' + \alpha''\bar{U}' ,$$

$$\eta_{12}\bar{U}'' = (\eta_{12}\bar{U}')' - \eta'_{12}\bar{U}' , \tag{7.17}$$

we see that the expression $(7.11)_1$ for g_1 may be rewritten in the form

$$g_1 = g_{11}\bar{U}' + g_{12}', \tag{7.18}$$

where

$$\begin{aligned} g_{11} &= -\alpha'' - 2\eta_{12}' + 4\{1+(\lambda^2-1)\bar{A}\}\Omega^2\alpha + 2\eta_{11} + 4\eta_{22} - 4\bar{\eta}_{21} , \\ g_{12} &= -\alpha\bar{U}'' + (\alpha' + 2\eta_{12})\bar{U}' + 4\bar{\eta}_{21}\bar{U} . \end{aligned} \tag{7.19}$$

The expression $(7.19)_2$ may be rewritten (cf.$(6.12)_1$) in the form

$$\begin{aligned} g_{12} &= -\alpha\{\bar{U}'' + 4\lambda^2\Omega^2\bar{U} + (-1)^n\lambda_3^{\frac{1}{2}}\Phi_1\} \\ &\quad + (\alpha' + 2\eta_{12})\bar{U}' + 4(\lambda^2\Omega^2\alpha + \bar{\eta}_{21})\bar{U} + (-1)^n\lambda_3^{\frac{1}{2}}\alpha\Phi_1 . \end{aligned} \tag{7.20}$$

Using $(6.12)_1$, $(6.13)_1$, $(7.6)_4$, (7.16), (B11), $(B13)_1$ and (3.20), we obtain from (7.20)

$$\begin{aligned} \int_{-\ell_2}^{\ell_2} g_{12}' d\xi_2 &= [\{7\lambda^2+1+(\lambda^2-1)\bar{A}\}\Omega^2 UU'\bar{U}' \\ &\quad -4\Omega^2\{(\lambda^2+1-\bar{A})U'^2 - \lambda^2\bar{A}\Omega^2U^2\}\bar{U}]_{-\ell_2}^{\ell_2} \\ &\quad -\frac{(-1)^n}{2\lambda_2}\Omega^2\{7\lambda^2+1+(\lambda^2-1)\bar{A}\}[UU'(U'^2+\lambda^2\Omega^2U^2)]_{-\ell_2}^{\ell_2} . \end{aligned} \tag{7.21}$$

In order to simplify the expression for g_{11}, we substitute from (7.6) and (7.16) in $(7.19)_1$ and employ (B7) and $(7.3)_2$ to obtain

$$\begin{aligned} g_{11} &= -3\beta(U''+\lambda^2\Omega^2U) + 6\beta'U' - \bar{A}\{6\lambda^2\Omega^2UU'' + 4U''^2 \\ &\quad + 2U'U''' + 3(\lambda^2+1)\lambda^2\Omega^4U^2 - 6(2\lambda^2+1)\Omega^2U'^2\} \\ &\quad + 8(\lambda-1)^3B\Omega^2U'^2 , \end{aligned} \tag{7.22}$$

where β is given by $(B7)_4$.

From $(7.11)_2$ we obtain, after a lengthy calculation in which equations $(B2)_2$, $(B3)_{1,5}$, $(B4)_2$, (B.5), (6.9), $(6.10)_1$, (6.14), $(7.3)_2$, $(7.5)_2$, and $(7.6)_4$ are employed,

$$g_2 = \frac{1}{2}(g_{22}-g'_{21}) \ , \tag{7.23}$$

where

$$g_{21} = \alpha(\alpha'-2\beta'U) + 2\bar{A}\alpha U'(U''+\lambda^2\Omega^2U) \ ,$$

and (7.24)

$$\begin{aligned} g_{22} = \alpha'^2 &+ 4\Omega^2\alpha^2\{1+(\lambda^2-1)\bar{A}\} + 4\alpha(\beta'U'-\beta''U) \\ &+ 16\beta'U'(UU')' + 8\lambda^2\Omega^2\{(UU')'\}^2 \\ &+ 4\bar{A}\{\alpha[3(\lambda^2+1)\Omega^2U'^2-U''^2-\lambda^2\Omega^4U^2-2\lambda^2\Omega^2\alpha] \\ &+ \alpha'U'(U''+\lambda^2\Omega^2U) + 8\lambda^2\Omega^2U'^2(UU')'\} \\ &+ 16(\lambda-1)^3B\Omega^2\alpha U'^2 \ . \end{aligned}$$

With $(3.20)_1$, $(7.24)_1$ and (B12), equation (7.23) yields

$$\begin{aligned} \int_{-\ell_2}^{\ell_2} g_2 d\xi_2 = &-\frac{1}{2}\Omega^2(\lambda^2-1)(1-\bar{A})[UU'(U'^2+\lambda^2\Omega^2U^2)]_{-\ell_2}^{\ell_2} \\ &+ \frac{1}{2}\int_{-\ell_2}^{\ell_2} g_{22}d\xi_2 \ . \end{aligned} \tag{7.25}$$

We now substitute in (7.10) from (7.11), (7.18), (7.21), (7.25) and (6.21) to obtain

$$\bar{G}[\underset{\sim}{u}] = 4\ell_1\ell_2\ell_3\lambda_3k_1(G_1-G_2+G_3-G_4) \ , \tag{7.26}$$

where

$$G_1 = \frac{1}{32\lambda^3\Omega^2\ell_2} \int_{-\ell_2}^{\ell_2} [g_{22}+(-1)^n \frac{2\lambda^{\frac{1}{2}}}{\lambda_3^{\frac{1}{2}}} g_{11}\bar{U}'+\frac{4}{\Omega^2} h]d\xi_2 \ ,$$

$$G_2 = \frac{1}{2\lambda\ell_2} [UU'\{U'^2+\lambda^2\Omega^2U^2\}]_{\xi_2=\ell_2} \ ,$$

$$G_3 = \frac{(-1)^n}{8\lambda^{5/2}\lambda_3^{\frac{1}{2}}\ell_2} [\{7\lambda^2+1+(\lambda^2-1)\bar{A}\}UU'\bar{U}' \tag{7.27}$$

$$-4\bar{U}\{U'^2(\lambda^2+1-\bar{A})-\bar{A}\lambda^2\Omega^2U^2\}]_{\xi_2=\ell_2} \ ,$$

$$G_4 = \frac{2}{\lambda\Delta} (a_1b_2^2+a_2b_1^2-2ab_1b_2) \ ,$$

where Δ is defined in (6.20) and the a's and b's in (6.16), $(6.17)_2$ and (6.18).

8. Some asymptotic expansions

In this and the following sections, we suppose that the plate is thin and calculate $\bar{G}[\bar{\underset{\sim}{u}}]$ for the case when the bifurcations is of the flexural type.

It has been shown in [3] that if the value of η defined by (cf.(3.15))

$$\eta = \Omega \ell_2 = n\pi \ell_2/(2\ell_1) \tag{8.1}$$

is small, then the critical value of λ at which a flexural bifurcation* can occur is given by

$$\lambda - 1 = \frac{2}{3}\eta^2 + \frac{16}{45}\eta^4 + O(\eta^6) \ . \tag{8.2}$$

We now introduce the notation

$$\begin{aligned} J_\alpha = I_\alpha\big|_{\lambda=1}\ , \quad W_\alpha^{(0)} = W_\alpha\big|_{\lambda=1}\ , \quad W_{\alpha\beta}^{(0)} = W_{\alpha\beta}\big|_{\lambda=1} \quad \text{etc.,} \\ k_\alpha^{(0)} = k_\alpha\big|_{\lambda=1}\ , \quad k_{\alpha\beta}^{(0)} = k_{\alpha\beta}\big|_{\lambda=1} \quad (\alpha,\beta=1,2)\ , \end{aligned} \tag{8.3}$$

and note, from $(2.9)_{1,2}$ and $(2.10)_1$ that

$$\begin{aligned} I_1 - J_1 = \lambda_3^{-2}(I_2 - J_2) = \lambda_3^{-1}(\lambda-1)^2\{1-(\lambda-1) + (\lambda-1)^2 \\ -(\lambda-1)^3 + O(\lambda-1)^4\}\ . \end{aligned} \tag{8.4}$$

---

* The integer n in (8.1) is the number of half-wavelengths along the 1-direction in the flexural mode under consideration. If the lowest-order mode is not suppressed by means of some type of passive constraint, no value of n greater than unity is relevant.

With (8.3) and (8.4), Taylor's theorem yields the following approximate expressions for W_α, $W_{\alpha\beta}$, $W_{\alpha\beta\gamma}$, $W_{\alpha\beta\gamma\delta}$:

$$
\begin{aligned}
W_\alpha = W_\alpha^{(0)} &+ \lambda_3^{-1}(\lambda-1)^2[1-(\lambda-1) + (\lambda-1)^2](W_{1\alpha}^{(0)}+\lambda_3^2 W_{2\alpha}^{(0)}) \\
&+ \tfrac{1}{2}\lambda_3^{-2}(\lambda-1)^4(W_{11\alpha}^{(0)}+2\lambda_3^2 W_{12\alpha}^{(0)}+\lambda_3^4 W_{22\alpha}^{(0)}) \\
&+ O(\lambda-1)^5 ,
\end{aligned} \tag{8.5}
$$

$$
\begin{aligned}
W_{\alpha\beta} &= W_{\alpha\beta}^{(0)}+\lambda_3^{-1}(\lambda-1)^2[1-(\lambda-1)](W_{1\alpha\beta}^{(0)}+\lambda_3^2 W_{2\alpha\beta}^{(0)})+O(\lambda-1)^4 , \\
W_{\alpha\beta\gamma} &= W_{\alpha\beta\gamma}^{(0)}+\lambda_3^{-1}(\lambda-1)^2(W_{1\alpha\beta\gamma}^{(0)}+\lambda_3^2 W_{2\alpha\beta\gamma}^{(0)})+O(\lambda-1)^3 , \\
W_{\alpha\beta\gamma\delta} &= W_{\alpha\beta\gamma\delta}^{(0)}+ O(\lambda-1)^2 .
\end{aligned}
$$

From (8.3), $(3.2)_{1,2}$ and $(A7)_1$ we have

$$
\begin{aligned}
k_1^{(0)} &= W_1^{(0)}+\lambda_3^2 W_2^{(0)} , \quad k_2^{(0)}=2(W_{11}^{(0)}+2\lambda_3^2 W_{12}^{(0)}+\lambda_3^4 W_{22}^{(0)}) , \\
k_3^{(0)} &= \tfrac{1}{6}(W_{111}^{(0)}+3\lambda_3^2 W_{112}^{(0)}+3\lambda_3^4 W_{122}^{(0)}+\lambda_3^6 W_{222}^{(0)}) .
\end{aligned} \tag{8.6}
$$

With (8.5) and (8.6), equations $(3.2)_{1,2}$ and $(A7)_{1,4}$ yield

$$
\begin{aligned}
k_1 &= k_1^{(0)}+\tfrac{1}{2}\lambda_3^{-1}k_2^{(0)}(\lambda-1)^2[1-(\lambda-1)+(\lambda-1)^2]+3\lambda_3^{-2}k_3^{(0)}(\lambda-1)^4 \\
&\qquad + O(\lambda-1)^5 , \\
k_2 &= k_2^{(0)}+12\lambda_3^{-1}k_3^{(0)}(\lambda-1)^2[1-(\lambda-1)]+O(\lambda-1)^4 , \\
k_3 &= k_3^{(0)}+4\lambda_3^{-1}k_4^{(0)}(\lambda-1)^2+O(\lambda-1)^3, \quad k_4=k_4^{(0)} + O(\lambda-1)^2 .
\end{aligned} \tag{8.7}
$$

From (7.8), $(B4)_2$, and (B5), we obtain with (8.7)

$$
\begin{aligned}
A &= A^{(0)}+\{\tfrac{1}{8}A^{(0)}(2-A^{(0)}) + B^{(0)}\}(\lambda-1)^2[1-(\lambda-1)] + O(\lambda-1)^4 , \\
B &= B^{(0)}-\tfrac{1}{2}B^{(0)}(\lambda-1) + O(\lambda-1)^2, \quad C = C^{(0)} + O(\lambda-1) ,
\end{aligned} \tag{8.8}
$$

where, from $(B4)_2$, (B5) and $(A7)_3$,

$$A^{(0)} = A\big|_{\lambda=1} = 4k_2^{(0)}/(\lambda_3 k_1^{(0)}) \ ,$$

$$B^{(0)} = B\big|_{\lambda=1} = 48k_3^{(0)}/(\lambda_3^2 k_1^{(0)}), \tag{8.9}$$

$$C^{(0)}=C\big|_{\lambda=1}= \frac{32}{\lambda_3^3 k_1^{(0)}} (W_{1111}^{(0)}+4\lambda_3^2 W_{1112}^{(0)}+6\lambda_3^4 W_{1122}^{(0)}+4\lambda_3^6 W_{1222}^{(0)} + \lambda_3^8 W_{2222}^{(0)}) \ .$$

9. Asymptotic first-order solution

We define the dimensionless thickness coordinate t by

$$t = \xi_2/\ell_2 \tag{9.1}$$

and write

$$V(t) = U(\xi_2) \ . \tag{9.2}$$

Then, noting from (3.15) and (8.1) that

$$\frac{1}{\Omega}\frac{d}{d\xi_2} = \frac{1}{\eta}\frac{d}{dt} \ , \tag{9.3}$$

and using the expression $(8.8)_1$ for $A^{(0)}$, we can rewrite the differential equation (3.19) as

$$\frac{1}{\eta^4}\frac{d^4V}{dt^4} - \{2+2(\lambda-1) + (\lambda-1)^2(1+A^{(0)}) + O(\lambda-1)^4\}\frac{1}{\eta^2}\frac{d^2V}{dt^2} + \{1+2(\lambda-1) + (\lambda-1)^2\}V = 0 \ , \tag{9.4}$$

and the boundary conditions (3.20) as

$$\begin{gathered}\frac{1}{\eta^2}\frac{d^2V}{dt^2} + \{1+2(\lambda-1) + (\lambda-1)^2\}V = 0 \ , \\ \frac{1}{\eta^3}\frac{d^3V}{dt^3} - \{3+4(\lambda-1)+(\lambda-1)^2(2+A^{(0)}) + O(\lambda-1)^4\}\frac{1}{\eta}\frac{dV}{dt} = 0 \ ,\end{gathered} \tag{9.5}$$

when $t = \pm 1$.

With the asymptotic expression for $\lambda-1$ given in (8.2), equation (9.4) can be rewritten as

$$\frac{1}{\eta^4}\frac{d^4V}{dt^4} - \{2 + \frac{4}{3}\eta^2 + \frac{4}{9}\eta^4(\frac{13}{5} + A^{(0)}) + O(\eta^6)\}\frac{1}{\eta^2}\frac{d^2V}{dt^2}$$

$$+ \{1 + \frac{4}{3}\eta^2 + \frac{52}{45}\eta^4 + O(\eta^6)\}V = 0 \qquad (9.6)$$

and the boundary conditions (9.5) can be rewritten as

$$\frac{1}{\eta^2}\frac{d^2V}{dt^2} + \{1 + \frac{4}{3}\eta^2 + \frac{52}{45}\eta^4 + O(\eta^6)\}V = 0 ,$$

$$\frac{1}{\eta^3}\frac{d^3V}{dt^3} - \{3 + \frac{8}{3}\eta^2 + \frac{4}{9}\eta^4(\frac{26}{5} + A^{(0)}) + O(\eta^6)\}\frac{1}{\eta}\frac{dV}{dt} = 0 \qquad (9.7)$$

on $t = \pm 1$. It can easily be verified that the solution of (9.6), with the boundary conditions (9.7), is given by*

$$V = 1 - \frac{1}{2}\eta^2 t^2 + \eta^4(\frac{1}{3}t^2 - \frac{1}{8}t^4) + \eta^6(\frac{4}{45}t^2 - \frac{1}{18}t^4 - \frac{1}{144}t^6) + O(\eta^8) . \qquad (9.8)$$

We note that to order η^6 , V is independent of both λ_3 and the form of the strain-energy function W .

---

* We employ the normalizing condition $V(0) = 1$.

10. Asymptotic second-order solution

We define $\bar{V}(t)$ by

$$\bar{U}(\xi_2) = (-1)^n \lambda_3^{\frac{1}{2}} \bar{V}(t) \ , \tag{10.1}$$

where t is defined by (9.1). Then, with (9.3), (8.2) and $(8.8)_1$, we can rewrite the differential equation (6.8) as

$$\frac{1}{\eta^4}\frac{d^4\bar{V}}{dt^4} - 4[2+\tfrac{4}{3}\eta^2 + O(\eta^4)]\frac{1}{\eta^2}\frac{d^2\bar{V}}{dt^2} + 16[1+\tfrac{4}{3}\eta^2 + O(\eta^4)]\bar{V} = -\frac{\Omega}{\eta}\frac{d\bar{\Phi}}{dt} \tag{10.2}$$

and the boundary conditions (6.12) as

$$\begin{aligned} &\frac{1}{\eta^2}\frac{d^2\bar{V}}{dt^2} + 4[1+\tfrac{4}{3}\eta^2 + \tfrac{52}{45}\eta^4 + O(\eta^6)]\bar{V} = -\bar{\Phi}_1 \ , \\ &\frac{1}{\eta^3}\frac{d^3\bar{V}}{dt^3} - 4[3+\tfrac{8}{3}\eta^2 + \tfrac{4}{9}\eta^4(\tfrac{26}{5} + A^{(0)}) + O(\eta^6)]\frac{1}{\eta}\frac{d\bar{V}}{dt} = \bar{\Phi}_2 \ , \end{aligned} \tag{10.3}$$

when $t = \pm 1$, where

$$\bar{\Phi}(t) = \Omega^{-4}\Phi(\xi_2) \ , \quad \bar{\Phi}_1(t) = \Omega^{-2}\Phi_1(\xi_2) \ , \quad \bar{\Phi}_2(t) = \Omega^{-3}\Phi_2(\xi_2) \ , \tag{10.4}$$

and $\Phi(\xi_2)$, $\Phi_1(\xi_2)$ and $\Phi_2(\xi_2)$ are given by (B11) and (6.13).

With (9.1), (8.2), (8.8) and (B5) we obtain from (10.4), (B11) and (B14)

$$-\frac{\Omega}{\eta}\frac{d\bar{\Phi}}{dt} = 4\Omega\eta^3 t(2+A^{(0)}) + O(\eta^5) \ ,$$
$$-\bar{\Phi}_1\Big|_{t=\pm 1} = \pm 2\Omega\eta[2+\tfrac{1}{3}\eta^2 + \eta^4(\tfrac{13}{15} + \tfrac{1}{9}A^{(0)}) + O(\eta^6)] \ , \qquad (10.5)$$
$$\bar{\Phi}_2\Big|_{t=\pm 1} = 2\Omega\eta^2[1+\eta^2(2+\tfrac{11}{9}A^{(0)}) + O(\eta^4)] \ .$$

With these expressions introduced on the right-hand sides of (10.2) and (10.3), it can easily be verified that the solution of (10.2), subject to the boundary conditions (10.3), is

$$\bar{V} = \Omega\eta t\{\tfrac{1}{4} + \eta^2(-\tfrac{7}{6}+\tfrac{1}{2}t^2) + \eta^4(\tfrac{23}{72} - \tfrac{1}{12}A^{(0)} - \tfrac{2}{9}t^2 + \tfrac{1}{6}t^4)$$
$$+ \eta^6[D-(\tfrac{23}{60} + \tfrac{1}{54}A^{(0)})t^2 + (\tfrac{2}{9} + \tfrac{1}{30}A^{(0)})t^4 + \tfrac{1}{45}t^6] + O(\eta^9) \ , \qquad (10.6)$$

where D is a constant which could be evaluated if the calculations were carried out to a higher order in η. However, the value of D will not be required for the calculation of $\bar{G}[\underset{\sim}{u}]$ which is the main object of the present paper.

It is seen from (10.6) that $\bar{V}$ - and hence, from (10.1), $\bar{U}$ - depends on the form of the strain-energy function W only through $A^{(0)}$, defined in $(8.9)_1$. From $(8.6)_1$ and $(8.9)_1$, we see that

$$A^{(0)} = \frac{8(W_{11}^{(0)}+2\lambda_3^2W_{12}^{(0)}+\lambda_3^4W_{22}^{(0)})}{\lambda_3(W_1^{(0)}+\lambda_3^2W_2^{(0)})} \ . \qquad (10.7)$$

11. The asymptotic expression for $\bar{G}[\bar{\underset{\sim}{u}}]$

In order to calculate $\bar{G}[\bar{\underset{\sim}{u}}]$ in the asymptotic case when $\ell_2/\ell_1 << 1$, we use the formulae (7.26) and (7.27) and introduce into the latter the relations (9.1), (9.2), (9.3) and (10.1). Then, we introduce, in the expressions for G_1, G_2, G_3 and G_4 so obtained, the expressions (9.8) and (10.6) for V and $\bar{V}$ and systematically neglect terms of higher degree than the fourth in η.

Calculation of G_1

Following this procedure we obtain from $(7.27)_1$,

$$G_1 = \frac{\Omega^4}{32\lambda^3} \int_{-1}^{1} (\hat{g}_{22} + \frac{2\lambda^{\frac{1}{2}}}{\Omega\eta} \hat{g}_{11} \frac{d\bar{V}}{dt} + 4\hat{h})dt \ , \qquad (11.1)$$

where $\hat{g}_{11}$, $\hat{g}_{22}$ and $\hat{h}$ are defined by

$$\hat{g}_{11} = \Omega^{-4} g_{11} \ , \quad \hat{g}_{22} = \Omega^{-6} g_{22} \ , \quad \hat{h} = \Omega^{-8} h \ , \qquad (11.2)$$

and g_{11}, g_{22} and h are defined by equations (7.22), $(7.24)_2$ and $(7.8)_1$. With (9.2), (9.3) and (10.1) and the notation

$$\hat{\alpha} = \Omega^{-2}\alpha \ , \ \hat{\beta} = \Omega^{-2}\beta \ , \ \hat{\kappa}_1 = \Omega^{-4}\kappa_1 \ , \ \hat{\kappa}_2 = \Omega^{-4}\kappa_2 \ , \qquad (11.3)$$

we obtain from these equations

$$\hat{g}_{11} = -3\hat{\beta}(\frac{1}{\eta^2}\frac{d^2V}{dt^2} + \lambda^2 V) + \frac{6}{\eta^2}\frac{d\hat{\beta}}{dt}\frac{dV}{dt}$$
$$-\bar{A}\left\{\frac{6\lambda^2}{\eta^2} V \frac{d^2V}{dt^2} + \frac{4}{\eta^4}(\frac{d^2V}{dt^2})^2 + \frac{2}{\eta^4}\frac{dV}{dt}\frac{d^3V}{dt^3}\right.$$

$$+ 3(\lambda^2+1)\lambda^2 V^2 - \frac{6}{\eta^2}(2\lambda^2+1)(\frac{dV}{dt})^2\Big\}$$

$$+ \frac{8}{\eta^2}(\lambda-1)^3 B(\frac{dV}{dt})^2 ,$$

$$\begin{aligned}
\hat{g}_{22} = {} & \frac{1}{\eta^2}(\frac{d\hat{\alpha}}{dt})^2 + 4\hat{\alpha}^2\Big\{1+(\lambda^2-1)\bar{A}\Big\} + \frac{4\hat{\alpha}}{\eta^2}(\frac{dV}{dt}\frac{d\hat{\beta}}{dt} - V\frac{d^2\hat{\beta}}{dt^2}) \\
& + \frac{16}{\eta^4}\frac{d\hat{\beta}}{dt}\frac{dV}{dt}\frac{d}{dt}(V\frac{dV}{dt}) + \frac{8\lambda^2}{\eta^4}\Big\{\frac{d}{dt}(V\frac{dV}{dt})\Big\}^2 \\
& + 4\bar{A}\Big\{\hat{\alpha}[\frac{3}{\eta^2}(\lambda^2+1)(\frac{dV}{dt})^2 - \frac{1}{\eta^4}(\frac{d^2V}{dt^2})^2 - \lambda^2V^2 - 2\lambda^2\hat{\alpha}] \\
& + \frac{1}{\eta^4}\frac{d\hat{\alpha}}{dt}\frac{dV}{dt}(\frac{d^2V}{dt^2} + \lambda^2\eta^2V) + \frac{8\lambda^2}{\eta^4}(\frac{dV}{dt})^2\frac{d}{dt}(V\frac{dV}{dt})\Big\} \\
& + \frac{16}{\eta^2}(\lambda-1)^3 B\hat{\alpha}(\frac{dV}{dt})^2 , \\
\hat{h} = {} & \frac{\bar{A}}{4(\lambda^2-1)}(2\hat{\kappa}_1^2+\hat{\kappa}_2^2) + \frac{2(\lambda-1)^2}{\eta^2(\lambda+1)}B(\frac{dV}{dt})^2(2\hat{\kappa}_1+\hat{\kappa}_2) \\
& + \frac{(\lambda-1)^4}{\eta^4}C(\frac{dV}{dt})^4 .
\end{aligned} \tag{11.4}$$

From (11.3) and the expressions (3.25), (3.7), (B.5) and (7.3), we obtain, with (9.2), (9.3) and (10.1),

$$\begin{aligned}
\hat{\alpha} &= \frac{1}{\eta^2}\Big\{V\frac{d^2V}{dt^2} - (\frac{dV}{dt})^2\Big\} , \\
\hat{\beta} &= \frac{1}{\eta^2}\frac{d^2V}{dt^2} - \Big\{1+(\lambda^2-1)\bar{A}\Big\}V , \\
\hat{\kappa}_1 &= \frac{\lambda^2+1}{\eta^2}(\frac{dV}{dt})^2 + \frac{1}{\eta^4}(\frac{d^2V}{dt^2})^2 + \lambda^2V^2 + \frac{2\lambda^2}{\eta^2}\frac{d}{dt}(V\frac{dV}{dt}) , \\
\hat{\kappa}_2 &= \frac{\lambda^2+1}{\eta^2}(\frac{dV}{dt})^2 - \frac{1}{\eta^4}(\frac{d^2V}{dt^2})^2 - \lambda^2V^2 - 2\lambda^2\hat{\alpha} .
\end{aligned} \tag{11.5}$$

From (B5), $(8.8)_1$ and (8.2), we obtain the following asymptotic expression for $\bar{A}$:

$$\bar{A} = A^{(0)}(\tfrac{1}{3}\eta^2 + \tfrac{1}{15}\eta^4) + O(\eta^6) \ . \tag{11.6}$$

We now substitute in (11.5) the asymptotic expressions for V, λ and $\bar{A}$, given in (9.8), (8.2) and (11.6) respectively, and obtain the following asymptotic expressions for $\hat{\alpha}$, $\hat{\beta}$, $\hat{\kappa}_1$ and $\hat{\kappa}_2$:

$$\begin{aligned}
\hat{\alpha} &= -1+\eta^2(\tfrac{2}{3}-2t^2) + \eta^4(\tfrac{8}{45} - \tfrac{1}{3}t^4) + O(\eta^6) \ , \\
\hat{\beta} &= -2+\eta^2(\tfrac{2}{3} - t^2) + \eta^4(\tfrac{8}{45} - \tfrac{4}{9}A^{(0)} - t^2 - \tfrac{1}{12}t^4) + O(\eta^6) \ , \\
\hat{\kappa}_1 &= \eta^2(-\tfrac{4}{3} + 4t^2) + \eta^4(\tfrac{16}{15} - 8t^2 + 8t^4) + O(\eta^6) \ , \\
\hat{\kappa}_2 &= \eta^2(\tfrac{4}{3} + 4t^2) + \eta^4(-\tfrac{16}{15} + \tfrac{16}{3}t^2) + O(\eta^6) \ .
\end{aligned} \tag{11.7}$$

With (9.8), (8.2), (11.6) and (11.7), equations (11.4) yield

$$\begin{aligned}
\hat{g}_{11} &= \eta^2(12-\tfrac{4}{3}A^{(0)}) + \eta^4\{4 - \tfrac{52}{45}A^{(0)} + 4(2+A^{(0)})t^2\} + O(\eta^6) \ , \\
\hat{g}_{22} &= 4-\eta^2(8+24t^2) - \eta^4\{\tfrac{32}{15} - \tfrac{64}{3}(1-A^{(0)})t^2 + 40t^4\} \\
&\qquad + O(\eta^6) \ , \\
\hat{h} &= \eta^4 A^{(0)}(\tfrac{1}{3} - \tfrac{2}{3}t^2 + 3t^4) + O(\eta^6) \ .
\end{aligned} \tag{11.8}$$

Substituting from (11.8) and (8.2) in (11.1) and carrying out the integration, we obtain the asymptotic expression for G_1:

$$G_1 = \Omega^4\{\tfrac{1}{4} - \eta^2(\tfrac{9}{8} + \tfrac{1}{24}A^{(0)}) + \eta^4(\tfrac{143}{180} - \tfrac{29}{360}A^{(0)}) + O(\eta^6)\}. \tag{11.9}$$

<u>Calculation of G_2</u>

With (9.2), (9.3) and (8.1) the expression $(7.27)_2$ for G_2 can be rewritten as

$$G_2 = \frac{\Omega^4}{2\lambda\eta^4} [V\frac{dV}{dt} \left\{(\frac{dV}{dt})^2 + \lambda^2\eta^2V^2\right\}]_{t=1} \ . \tag{11.10}$$

With the asymptotic expressions (8.2) and (9.8) for λ and V respectively, this yields

$$G_2 = -\Omega^4(\frac{1}{2} - \frac{16}{45}\eta^4) + O(\eta^6) \ . \tag{11.11}$$

Calculation of G_3

With (9.2), (9.3) and (10.1) the expression $(7.27)_3$ for G_3 can be rewritten as

$$G_3 = \frac{\Omega^3}{8\lambda^{5/2}\eta^3} [\{7\lambda^2+1+(\lambda^2-1)\bar{A}\}V\frac{dV}{dt}\frac{d\bar{V}}{dt} -4\{(\lambda^2+1-\bar{A})(\frac{dV}{dt})^2 - \bar{A}\lambda^2\eta^2V^2\}\bar{V}]_{t=1} \ . \tag{11.12}$$

We introduce the asymptotic expressions (9.8), (10.6) and (8.2) for V, $\bar{V}$ and λ respectively and obtain, with (11.6),

$$G_3 = \Omega^4\{-\frac{1}{4}+\eta^2(-\frac{7}{24} + \frac{1}{24}A^{(0)}) + \eta^4(\frac{25}{36} - \frac{17}{360}A^{(0)}) + O(\eta^6)\} \ . \tag{11.13}$$

Calculation of G_4

With (9.1), (9.2) and (9.3), we can rewrite the expressions (B16) for b_1 and b_2 as

$$b_1 = -\frac{\Omega^2}{2\lambda}\int_{-1}^{1}\Big[\frac{1}{\eta^4}\Big(\frac{d^2V}{dt^2}\Big)^2 + \frac{1}{\eta^2}\Big(\frac{dV}{dt}\Big)^2\{1-\frac{1}{2}(\lambda^2+3)\bar{A}$$
$$+\frac{1}{4}(\lambda^2-1)\bar{A}^2 - (\lambda-1)^3B\}\Big]dt\ ,$$
$$(11.14)$$
$$b_2 = -\frac{\Omega^2}{4\lambda}\int_{-1}^{1}\Big[\frac{1}{\eta^4}\Big(\frac{d^2V}{dt^2}\Big)^2 + \frac{1}{\eta^2}\Big(\frac{dV}{dt}\Big)^2\{1$$
$$-[\lambda^2+1+(\lambda^2-1)\bar{c}]\bar{A}+(\lambda^2-1)^2\bar{d}\}\Big]dt\ ,$$

where $\bar{c}$ and $\bar{d}$ are defined in (B.17). We introduce the asymptotic expressions for λ, V and $\bar{A}$ given in (8.2), (9.8) and (11.6) and obtain

$$b_1 = 2b_2 = -\Omega^2\{1-\frac{2}{3}\eta^2 + \eta^4\Big(\frac{11}{45} - \frac{2}{9}A^{(0)}\Big) + O(\eta^6)\}\ . \qquad (11.15)$$

Substituting from (11.15) in $(7.27)_4$ and using (6.20), we obtain

$$G_4 = 2\Psi\Omega^4\{1-2\eta^2 + \eta^4\Big(\frac{86}{45} - \frac{4}{9}A^{(0)}\Big) + O(\eta^6)\}\ , \qquad (11.16)$$

where

$$\Psi = \frac{a_1+4a_2-4a}{4(a_1a_2-a^2)} \qquad (11.17)$$

and a_1, a_2 and a are defined in (6.16). We note that they depend explicitly on λ_3 .

In order to simplify our calculations of the asymptotic expressions for a_1, a_2 and a , we will restrict them to the case when $\lambda_3 = 1$. Then introducing (8.2), $(8.5)_{1,2}$, $(8.7)_1$ and (8.8) into (6.16), and taking $\lambda_3 = 1$, we obtain

$$\begin{aligned} a_1 &= 4\{1+\eta^2+\eta^4(\tfrac{13}{15}+\tfrac{1}{9}A_0) + O(\eta^6)\} , \\ a_2 &= 4\{1+\eta^2(\tfrac{1}{2}+\tfrac{2}{3}\zeta) \\ &\qquad + \eta^4(\tfrac{4}{15}+\tfrac{31}{45}\zeta+\tfrac{1}{36}A_0) + O(\eta^6)\}, \\ a &= 2\{1+\tfrac{4}{3}\zeta\eta^2+\eta^4(\tfrac{52}{45}\zeta+\tfrac{1}{9}A_0) + O(\eta^6)\} , \end{aligned} \tag{11.18}$$

where

$$A_0 = A^{(0)}\Big|_{\lambda_3=1} \quad \text{and} \quad \zeta = \frac{W_1^{(0)}}{W_1^{(0)}+W_2^{(0)}}\Bigg|_{\lambda_3=1} \tag{11.19}$$

With (11.18) we obtain from (11.17)

$$\Psi = \frac{1}{12}\{3-3\eta^2+\eta^4(\tfrac{7}{5}-\tfrac{8}{3}\zeta+\tfrac{16}{9}\zeta^2-\tfrac{1}{3}A_0) + O(\eta^6)\}. \tag{11.20}$$

Then, introducing (11.20) into (11.16), we have

$$G_4 = \Omega^4\{\tfrac{1}{2}-\tfrac{3}{2}\eta^2+\eta^4(\tfrac{197}{90}-\tfrac{5}{18}A_0-\tfrac{4}{9}\zeta+\tfrac{8}{27}\zeta^2) + O(\eta^6)\} \tag{11.21}$$

<u>Calculation of $\bar{G}[\underset{\sim}{u}]$ when $\lambda_3=1$</u>

We now introduce into the expression (7.26) for $\bar{G}[\underset{\sim}{\bar{u}}]$, the expressions (11.9), (11.11), (11.13) and (11.21) for G_1, G_2, G_3 and G_4 and take $\lambda_3 = 1$. With $(8.7)_1$ and (8.2) we thus obtain

$$\bar{G}[\underset{\sim}{\bar{u}}] = \tfrac{1}{3}\ell_1\ell_2\ell_3K_0\Omega^4\eta^2\{1+\eta^2(-\tfrac{38}{3}+\tfrac{9}{5}A_0+\tfrac{16}{3}\zeta-\tfrac{32}{9}\zeta^2)\} + O(\eta^6) , \tag{11.22}$$

where

$$K_0 = k_1^{(0)}\Big|_{\lambda_3=1} = [W_1^{(0)} + W_2^{(0)}]_{\lambda_3=1} \ . \tag{11.23}$$

It is instructive to make the substitutions

$$\Omega = \frac{\eta}{\ell_2} \ , \ \delta = \frac{9}{5} A_0 + \frac{16}{9} \frac{W_2^{(0)}}{k_1^{(0)}} (1 - \frac{2W_2^{(0)}}{k_1^{(0)}}) \ , \tag{11.24}$$

in (11.22), which may then be rewritten as

$$\bar{G}[\bar{\underset{\sim}{u}}] = \frac{\ell_1 \ell_3 K_0 \eta^6}{3\ell_2^3} [1 - \eta^2 (\frac{98}{9} - \delta) + O(\eta^4)] \ . \tag{11.25}$$

If the material is neo-Hookean, $\delta = 0$ and equation (11.25) becomes

$$\bar{G}[\bar{\underset{\sim}{u}}] = \frac{\ell_1 \ell_3 K_0 \eta^6}{3\ell_2^3} [1 - \frac{98}{9}\eta^2 + O(\eta^4)] \ . \tag{11.26}$$

We may compare this result with that obtained in a previous paper [2] in which an analysis similar to that in the present paper was carried out for a neo-Hookean material with $K_0 = \frac{1}{2}$ and arbitrary λ_3 . It was found (see (7.19) in [2]) that

$$\bar{G}[\bar{\underset{\sim}{u}}] = \frac{\ell_1 \ell_3 \lambda_3 \eta^6}{6\ell_2^3} [1 - \frac{2}{3}(16 + \frac{1}{2+\lambda_3^3})\eta^2 + O(\eta^4)] \ . \tag{11.27}$$

By taking $\lambda_3 = 1$ in (11.27) and $K_0 = \frac{1}{2}$ in (11.26) we see that agreement is obtained.

Appendix A

In this section we shall prove the result expressed by equation (4.7).

We introduce the notation (cf. $(3.3)_1$)

$$\bar{k} = \lambda_1(1-\lambda^2)\bar{u}_{1,1} + (\lambda_3^2-\lambda_2^2)\bar{E} \tag{A1}$$

and the operator $\kappa[\hat{\underset{\sim}{u}},\bar{\underset{\sim}{u}}]$ defined by

$$\kappa[\varepsilon\hat{\underset{\sim}{u}}+\varepsilon^2\bar{\underset{\sim}{u}}] = \varepsilon^2\kappa[\hat{\underset{\sim}{u}}] + 2\varepsilon^3\kappa[\hat{\underset{\sim}{u}},\bar{\underset{\sim}{u}}] + \varepsilon^4\kappa[\bar{\underset{\sim}{u}}] \ . \tag{A2}$$

With $(2.10)_2$, we obtain

$$\kappa[\hat{\underset{\sim}{u}},\bar{\underset{\sim}{u}}] = \hat{u}_{\alpha,\beta}\bar{u}_{\alpha,\beta}+\lambda(\hat{u}_{1,2}\bar{u}_{2,1}+\hat{u}_{2,1}\bar{u}_{1,2}-\hat{u}_{1,1}\bar{u}_{2,2} -\hat{u}_{2,2}\bar{u}_{1,1}) \ . \tag{A3}$$

With this notation and equations (4.3) and $(3.3)_{2,3}$, we obtain from (2.12), by neglecting terms of higher degree than the fourth in ε,

$$\begin{aligned} i &= 2\varepsilon\lambda_1(1-\lambda^2)\hat{u}_{1,1}+\varepsilon^2\{\kappa[\hat{\underset{\sim}{u}}] + 2\bar{k}\} + 2\varepsilon^3\kappa[\hat{\underset{\sim}{u}},\bar{\underset{\sim}{u}}] \\ &\qquad + \varepsilon^4\{\kappa[\bar{\underset{\sim}{u}}] + (\lambda_3^2+2\lambda_2^2)\bar{E}^2\} \ , \\ j &= 2\varepsilon^2\bar{E}\Lambda_2 + 4\varepsilon^3\bar{E}\lambda_1(1-\lambda^2)\hat{u}_{1,1} \\ &\qquad + \varepsilon^4\bar{E}\{2\kappa[\hat{\underset{\sim}{u}}] + 4\lambda_1(1-\lambda^2)\bar{u}_{1,1} + \Lambda_1\bar{E}\} \ . \end{aligned} \tag{A4}$$

Again to order ε^4, we obtain from (A4)

$$i^2 = 4\varepsilon^2\lambda_1^2(1-\lambda^2)^2(\hat{u}_{1,1})^2 + 4\varepsilon^3\lambda_1(1-\lambda^2)\hat{u}_{1,1}\{\kappa[\hat{\underset{\sim}{u}}]+2\bar{k}\}$$
$$+ \varepsilon^4\{(\kappa[\hat{\underset{\sim}{u}}])^2+8\lambda_1(1-\lambda^2)\hat{u}_{1,1}\kappa[\bar{\underset{\sim}{u}},\hat{\underset{\sim}{u}}] +4\kappa[\hat{\underset{\sim}{u}}]\bar{k}+4\bar{k}^2\} ,$$

$$i^3 = 8\varepsilon^3\lambda_1^3(1-\lambda^2)^3(\hat{u}_{1,1})^3$$
$$+ 12\varepsilon^4\lambda_1^2(1-\lambda^2)^2(\hat{u}_{1,1})^2\{\kappa[\hat{\underset{\sim}{u}}]+2\bar{k}\} ,$$

$$i^4 = 16\varepsilon^4\lambda_1^4(1-\lambda^2)^4(\hat{u}_{1,1})^4 , \tag{A5}$$

$$ij = 4\varepsilon^3\Lambda_2\lambda_1(1-\lambda^2)\hat{u}_{1,1}\bar{E} + 2\varepsilon^4\bar{E}\{\Lambda_2\kappa[\hat{\underset{\sim}{u}}]+2\Lambda_2\bar{k}$$
$$+ 4\lambda_1^2(1-\lambda^2)^2(\hat{u}_{1,1})^2\} ,$$

$$i^2j = 8\varepsilon^4\Lambda_2\lambda_1^2(1-\lambda^2)^2(\hat{u}_{1,1})^2\bar{E} ,$$

$$j^2 = 4\varepsilon^4\Lambda_2^2\bar{E}^2 .$$

Also, from (2.11) and (2.16) we obtain to order ε^4 ,

$$W^{(3)} = k_3 i^3 + k_{31} i^2 j , \quad W^{(4)} = k_4 i^4 , \tag{A6}$$

where

$$\begin{aligned} k_3 &= \tfrac{1}{6}(W_{111}+3\lambda_3^2W_{112}+3\lambda_3^4W_{122}+\lambda_3^6W_{222}) , \\ k_{31} &= \tfrac{1}{2}\lambda_3^2(W_{112}+2\lambda_3^2W_{122}+\lambda_3^4W_{222}) , \\ k_4 &= \tfrac{1}{24}(W_{1111}+4\lambda_3W_{1112}+6\lambda_3^4W_{1122}+4\lambda_3^6W_{1222}+\lambda_3^8W_{2222}) . \end{aligned} \tag{A7}$$

We now substitute from (4.3) in (2.22), and use the relations (3.1), (3.2), (3.3), $(3.21)_1$, (A2), (A5), to obtain, to order ε^4,

$$\begin{aligned} G[\underset{\sim}{u}] &= G[\varepsilon\hat{\underset{\sim}{u}}+\varepsilon^2\bar{\underset{\sim}{u}}] \\ &= \varepsilon^2G_2[\hat{\underset{\sim}{u}}] + 2\ell_3\varepsilon^3 \iint (g_1^{(3)}+g_2^{(3)}+\varepsilon g^{(4)})d\xi_1 d\xi_2 , \end{aligned} \tag{A8}$$

where

$$g_1^{(3)} = 2\{k_1\kappa[\hat{\underset{\sim}{u}},\bar{\underset{\sim}{u}}] + \lambda_1^2(1-\lambda^2)^2k_2\hat{u}_{1,1}\bar{u}_{1,1}\} ,$$

$$g_2^{(3)} = \lambda_1(1-\lambda^2)[2\{2\lambda_3^2W_2+(\lambda_3^2-\lambda_2^2)k_2+\Lambda_2k_{21}\}\bar{E}\hat{u}_{1,1} + k_2\hat{u}_{1,1}\kappa[\hat{\underset{\sim}{u}}] + 8\lambda_1^2(1-\lambda^2)^2k_3(\hat{u}_{1,1})^3] ,$$

$$\begin{aligned} g^{(4)} = {} & k_1\{\kappa[\bar{\underset{\sim}{u}}] + (\lambda_3^2+2\lambda_2^2)\bar{E}^2\} \\ & + \lambda_3^2W_2\{2\kappa[\hat{\underset{\sim}{u}}]\bar{E} + 4\lambda_1(1-\lambda^2)\bar{E}\bar{u}_{1,1}+\Lambda_1\bar{E}^2\} \\ & + k_2\{\tfrac{1}{4}(\kappa[\hat{\underset{\sim}{u}}])^2 + 2\lambda_1(1-\lambda^2)\hat{u}_{1,1}\kappa[\hat{\underset{\sim}{u}},\bar{\underset{\sim}{u}}]+\bar{k}\kappa[\hat{\underset{\sim}{u}}]+\bar{k}^2\} \\ & + k_{21}\{4\lambda_1^2(1-\lambda^2)^2(\hat{u}_{1,1})^2\bar{E} + \Lambda_2\kappa[\hat{\underset{\sim}{u}}]\bar{E} + 2\Lambda_2\overline{kE}\} \\ & + 2\lambda_3^4W_{22}\Lambda_2^2\bar{E}^2 + 12\lambda_1^2(1-\lambda^2)^2k_3(\hat{u}_{1,1})^2\{\kappa[\hat{\underset{\sim}{u}}] + 2\bar{k}\} \\ & + 8\lambda_1^2(1-\lambda^2)^2\Lambda_2k_{31}(\hat{u}_{1,1})^2\bar{E} \\ & + 16\lambda_1^4(1-\lambda^2)^4k_4(\hat{u}_{1,1})^4 . \end{aligned} \qquad (A9)$$

Appendix B

We substitute in (5.4) the expressions for $\hat{u}_1$, $\hat{u}_2$ and $\hat{p}$ given in (3.14) and obtain, with (3.16) and (3.24),

$$\begin{aligned}
\bar{F}_{11} &= \{k_1+\lambda_1^2(\lambda^2-1)^2k_2\}\bar{u}_{1,1}-\lambda k_1\bar{u}_{2,2}-\lambda_1^{-1}\bar{p}\\
&\quad -\lambda_1(\lambda^2-1)\{2\lambda_3^2W_2-(\lambda_2^2-\lambda_3^2)k_2+\Lambda_2k_{21}\}\bar{E}\\
&\quad -f_{11}-(-1)^n\phi_{11}\cos 2\Omega\xi_1 + \bar{\chi}\hat{u}_{1,1},\\
\bar{F}_{21} &= k_1(\bar{u}_{2,1}+\lambda\bar{u}_{1,2})-(-1)^n\phi_{21}\sin 2\Omega\xi_1 + \bar{\chi}\hat{u}_{2,1},\\
\bar{F}_{12} &= k_1(\bar{u}_{1,2}+\lambda\bar{u}_{2,1})-(-1)^n\phi_{12}\sin 2\Omega\xi_1 + \bar{\chi}\hat{u}_{1,2},\\
\bar{F}_{22} &= k_1(\bar{u}_{2,2}-\lambda\bar{u}_{1,1})-\lambda_2^{-1}\bar{p} - f_{22}-(-1)^n\phi_{22}\cos 2\Omega\xi_1+ \bar{\chi}\hat{u}_{2,2},\\
\bar{F}_{33} &= \{(\lambda_3^2+2\lambda_2^2)k_1+\lambda_3^2\Lambda_1W_2+(\lambda_2^2-\lambda_3^2)^2k_2\\
&\quad -2(\lambda_2^2-\lambda_3^2)\Lambda_2k_{21}+2\lambda_3^4\Lambda_2^2W_{22}\}\bar{E}-\bar{p}\\
&\quad -\lambda_1(\lambda^2-1)\{2\lambda_3^2W_2-(\lambda_2^2-\lambda_3^2)k_2+\Lambda_2k_{21}\}\bar{u}_{1,1}\\
&\quad -f_{33}-(-1)^n\phi_{33}\cos 2\Omega\xi_1,
\end{aligned} \tag{B1}$$

where the f's and ϕ's are functions of ξ_2 only defined by

$$\begin{aligned}
f_{11} &= (4\lambda\lambda_2\Omega^2)^{-1}\{2k_1\beta'U'\\
&\quad +\lambda_1^2(\lambda^2-1)k_2[3(\lambda^2+1)\Omega^2U'^2+U''^2+\lambda^2\Omega^4U^2+2\lambda^2\Omega^2(UU')']\\
&\quad +24\lambda_1^4(\lambda^2-1)^3k_3\Omega^2U'^2\},\\
f_{22} &= -(2\lambda^2\lambda_2\Omega^2)^{-1}\{k_1\beta'U'+2\lambda_1^2(\lambda^2-1)k_2\lambda^2\Omega^2U'^2\},\\
f_{33} &= -(2\lambda^2\Omega^2)^{-1}\{[\lambda_3^2W_2-\tfrac{1}{2}(\lambda_2^2-\lambda_3^2)k_2+\tfrac{1}{2}\Lambda_2k_{21}]\\
&\quad \times[(\lambda^2+1)\Omega^2U'^2+U''^2+\lambda^2\Omega^4U^2+2\lambda^2\Omega^2(UU')']\\
&\quad +2\lambda_1^2(\lambda^2-1)^2[k_{21}-6(\lambda_2^2-\lambda_3^2)k_3+2\Lambda_2k_{31}]\Omega^2U'^2\},
\end{aligned} \tag{B2}$$

and

$$\phi_{11} = (4\lambda\lambda_2\Omega^2)^{-1}\{2k_1\beta'U'+\lambda_1^2(\lambda^2-1)k_2[3(\lambda^2+1)\Omega^2U'^2$$
$$-U''^2-\lambda^2\Omega^4U^2-2\lambda^2\Omega^2\alpha] + 24\lambda_1^4(\lambda^2-1)^3k_3\Omega^2U'^2\} ,$$

$$\phi_{22} = f_{22} ,$$

$$\phi_{33} = -(2\lambda^2\Omega^2)^{-1}\{[\lambda_3^2W_2-\tfrac{1}{2}(\lambda_2^2-\lambda_3^2)k_2+\tfrac{1}{2}\Lambda_2k_{21}] \tag{B3}$$
$$\times[(\lambda^2+1)\Omega^2U'^2-U''^2-\lambda^2\Omega^4U^2-2\lambda^2\Omega^2\alpha]$$
$$+ 2\lambda_1^2(\lambda^2-1)^2[k_{21}-6(\lambda_2^2-\lambda_3^2)k_3+2\Lambda_2k_{31}]\Omega^2U'^2\} ,$$

$$\phi_{21} = (2\lambda^2\lambda_2\Omega^3)^{-1}\{k_1\beta'U''+\lambda_1^2(\lambda^2-1)k_2\lambda^2\Omega^2U'(U''+\Omega^2U)\} ,$$

$$\phi_{12} = (2\lambda\lambda_2\Omega)^{-1}\{k_1\beta'U+\lambda_1^2(\lambda^2-1)k_2U'(U''+\lambda^2\Omega^2U)\} .$$

We introduce the notation

$$\phi' = \frac{4\lambda^2\lambda_2\Omega^3}{k_1}\phi_{21} , \quad B = \frac{6\lambda_1^4(\lambda+1)^3k_3}{k_1} \tag{B4}$$

and (cf.(3.18))

$$\bar{A} = \frac{\lambda-1}{\lambda+1}A = \frac{(\lambda^2-1)k_2}{\lambda\lambda_3k_1} . \tag{B5}$$

Then, we obtain from $(B3)_{1,2,4,5}$, $(B2)_2$ and (3.17)

$$\phi_{11} = \frac{k_1}{4\lambda\lambda_2\Omega^2}\{2\beta'U'+\bar{A}[3(\lambda^2+1)\Omega^2U'^2-U''^2$$
$$-\lambda^2\Omega^4U^2-2\lambda^2\Omega^2\alpha] + 4(\lambda-1)^3B\Omega^2U'^2\} , \tag{B6}$$

$$\phi_{22} = -\frac{k_1}{2\lambda^2\lambda_2\Omega^2}(\beta'U'+2\bar{A}\lambda^2\Omega^2U'^2) ,$$

$$\phi_{12} = \frac{k_1}{2\lambda\lambda_2\Omega}\{\beta' U + \bar{A}U'(U'' + \lambda^2\Omega^2 U)\} \ ,$$

$$\phi' = [U''^2 - \{1+(\lambda^2-1)\bar{A}\}\Omega^2 U'^2 + \bar{A}\lambda^2\Omega^2(U'^2+\Omega^2U^2)]' \ .$$

We note from (3.17), (3.19) and (3.25), with the notation (B5), the relations

$$\begin{aligned}
\alpha &= UU''-U'^2 \ , \quad \alpha' = UU''' - U'U'' \ , \\
\alpha'' &= \Omega^2\{\lambda^2+1+(\lambda^2-1)\bar{A}\}(\alpha+U'^2) - (U''^2+\lambda^2\Omega^4U^2) \ , \\
\beta &= U''-\{1+(\lambda^2-1)\bar{A}\}\Omega^2 U \ , \ \beta' = U''' - \{1+(\lambda^2-1)\bar{A}\}\Omega^2U' \ , \\
\beta'' &= \lambda^2\Omega^2(U''-\Omega^2U) \ .
\end{aligned} \tag{B7}$$

With these relations, equations $(B6)_{2,3}$ yield

$$\begin{aligned}
\phi_{22}' &= -\frac{k_1}{4\lambda^2\lambda_2\Omega^2}[U''^2+\{\lambda^2-1+(3\lambda^2+1)\bar{A}\}\Omega^2U'^2-\lambda^2\Omega^4U^2]' \ , \\
\phi_{12}' &= \frac{k_1}{2\lambda\lambda_2\Omega}\{(1+\bar{A})(U'U'''+\lambda^2\Omega^2UU'')-\lambda^2\Omega^4U^2 \\
&\qquad + \bar{A}U''^2-(1-\bar{A})\Omega^2U'^2\} \ .
\end{aligned} \tag{B8}$$

We now substitute from $(B6)_1$ and $(B8)_2$ in the expression (6.9) for q and use the relations (B7) to obtain

$$\begin{aligned}
q = \frac{1}{2\lambda_2}\,[&\{\lambda^2-3+(\lambda^2+3)\bar{A}\}\Omega^2\alpha-(1-4\bar{A})U''^2 \\
&+\{\lambda^2+3-(\lambda^2+9)\bar{A}-8(\lambda-1)^3B\}\Omega^2U'^2 \\
&-(3-2\bar{A})\lambda^2\Omega^4U^2-2(1-\bar{A})U'U'''+2(1+\bar{A})\lambda^2\Omega^2UU''] \ .
\end{aligned} \tag{B9}$$

From (3.19) and (B5) it follows that

$$2(U'U''')' = [(U'')^2+\{\lambda^2+1+(\lambda^2-1)\bar{A}\}\Omega^2U'^2-\lambda^2\Omega^4U^2]' . \tag{B10}$$

Then, from (B9), (B10), $(B8)_1$, $(B6)_4$ and $(B4)_1$ we obtain the following expression for Φ defined by (6.9):

$$\begin{aligned}\Phi = \tfrac{1}{2}\lambda^{-\frac{1}{2}}[&3\{(\lambda^2-1) + (\lambda^2+1)\bar{A}\}\Omega^2\alpha \\ &-\{5(\lambda^2+1)\bar{A}-(\lambda^2-1)\bar{A}^2+8(\lambda-1)^3B\}\Omega^2U'^2 \\ &+ 5\bar{A}(U''^2+\lambda^2\Omega^4U^2] .\end{aligned} \tag{B11}$$

From (3.20), (B5) and (B7) we obtain

$$\left.\begin{aligned}\alpha &= -(U'^2+\lambda^2\Omega^2U^2) , \\ \alpha' &= \{3\lambda^2+1+(\lambda^2-1)\bar{A}\}\Omega^2UU' , \\ \beta' &= 2\lambda^2\Omega^2U' .\end{aligned}\right\} \quad \text{on} \quad \xi_2 = \pm\ell_2 . \tag{B12}$$

With these relations and (3.20), we obtain from (B6) and (B9)

$$\begin{aligned}\phi_{12}(\pm\ell_2) &= \frac{\lambda\Omega k_1}{\lambda_2} UU' , \quad \phi_{22}(\pm\ell_2) = -\frac{k_1}{\lambda_2}(1+\bar{A})U'^2 , \\ q &= -\frac{\Omega^2}{2\lambda_2}[2\{2(\lambda^2-1)+4\bar{A}-(\lambda^2-1)\bar{A}^2+4(\lambda-1)^3B\}U'^2 \\ &\quad + \lambda^2\{4\lambda^2-(\lambda^2-1)\bar{A}\}\Omega^2U^2] .\end{aligned} \tag{B13}$$

Equations (B12) and (B13) yield the following expressions for Φ_1 and Φ_2 defined in (6.13):

$$\Phi_1 = \tfrac{1}{2}\Omega^2\lambda^{-\frac{1}{2}}\{7\lambda^2+1+(\lambda^2-1)\bar{A}\}UU' \;,$$

$$\Phi_2 = 2\Omega^2\lambda^{-\frac{1}{2}}[\{2\lambda^2-1+2(\lambda^2+1)\bar{A}-\tfrac{1}{2}(\lambda^2-1)\bar{A}^2$$
$$+\; 2(\lambda-1)^3B\}U'^2-\tfrac{1}{4}\lambda^2(\lambda^2-1)\bar{A}\Omega^2U^2] \;. \tag{B14}$$

From $(B7)_5$ and (3.19) we readily obtain, with (B5), the identities

$$\lambda^2\Omega^4U^2 = [U'(U''+\lambda^2\Omega^2U)-U\{U'''-[2\lambda^2+1+(\lambda^2-1)\bar{A}]\Omega^2U'\}]'$$
$$-2\lambda^2\Omega^2(UU')'-U''^2-\{\lambda^2+1+(\lambda^2-1)\bar{A}\}\Omega^2U'^2 \;, \tag{B15}$$

$$\lambda^2\Omega^2(UU')'+\beta'U'=\{U'(U''+\lambda^2\Omega^2U)\}'-U''^2-\{1+(\lambda^2-1)\bar{A}\}\Omega^2U'^2 \;.$$

We now substitute from $(B2)_{1,2,3}$ in (6.18) and $(6.17)_2$ and use (3.20), $(B4)_2$, (B5) and (B15) to obtain

$$b_1 = -\frac{1}{2\ell_2\lambda\Omega^2}\int_{-\ell_2}^{\ell_2} [U''^2+\Omega^2U'^2\{1-\tfrac{1}{2}(\lambda^2+3)\bar{A}$$
$$+\; \tfrac{1}{4}(\lambda^2-1)\bar{A}^2-(\lambda-1)^3B\}]d\xi_2 \;,$$

$$b_2 = -\frac{1}{4\ell_2\lambda\Omega^2}\int_{-\ell_2}^{\ell_2} [U''^2+\Omega^2U'^2\{1$$
$$-[(\lambda^2+1)+(\lambda^2-1)\bar{c}]\bar{A}+(\lambda^2-1)^2\bar{d}\}]d\xi_2 \;, \tag{B16}$$

where $\bar{c}$ and $\bar{d}$ are defined by

$$\bar{c} = \frac{1}{k_1}\,(\lambda_3^2W_2+\tfrac{1}{2}\Lambda_2k_{21}) - \tfrac{1}{2}\lambda\,\frac{\lambda-\lambda_3^3}{\lambda^2-1}\,\bar{A} \;,$$

$$\bar{d} = \frac{2}{\lambda\lambda_3k_1}\,(k_{21}+2\Lambda_2k_{31})-2\lambda\,\frac{\lambda-\lambda_3^3}{(\lambda+1)^3}\,B \;, \tag{B17}$$

with k_1 and k_{21} defined in (3.2), k_3 in (A7), and Λ_2 in (3.3).

Acknowledgement

The contribution of one of the authors (R.S.R.) to this work was supported by the Office of Naval Research under Contract No. N00014-76-C-0235. The contribution of the other author (K.N.S.) was supported in part by a grant from the National Science Foundation to Lehigh University.

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CAM-100-38	2. GOVT ACCESSION NO. AD-A114721	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Further Results on the Stability of a Thick Elastic Plate under Thrust		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) K.N. Sawyers and R.S. Rivlin		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0235
9. PERFORMING ORGANIZATION NAME AND ADDRESS Lehigh University Center for the Application of Mathematics Bethlehem, Pa. 18015		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA.		12. REPORT DATE April, 1982
		13. NUMBER OF PAGES 57
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Same as above		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution of this document is unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Buckling stability, finite elasticity, bifurcation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The stability of the critical state for flexural buckling of a plate of incompressible isotropic elastic material with arbitrary strain-energy function is studied. Most of the analysis is carried out without restriction on the magnitude of the aspect ratio of the plate. However, the final result is limited to the case when the aspect ratio is such that terms of fourth degree in it may be neglected in comparison with terms of zero degree.

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S/N 0102-014-6601

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